

CLASSICAL \mathcal{W} -ALGEBRAS AND DRINFELD-SOKOLOV BI-HAMILTONIAN SYSTEMS WITHIN THE THEORY OF POISSON VERTEX ALGEBRAS

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ABSTRACT. We provide a description of the Drinfeld-Sokolov Hamiltonian reduction for the construction of classical \mathcal{W} -algebras within the framework of Poisson vertex algebras. In this context, the gauge group action on the phase space is translated in terms of (the exponential of) a Lie conformal algebra action on the space of functions. Following the ideas of Drinfeld and Sokolov, we then establish under certain sufficient conditions the applicability of the Lenard-Magri scheme of integrability and the existence of the corresponding integrable hierarchy of bi-Hamiltonian equations.

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0. INTRODUCTION

In the seminal paper [DS85], Drinfeld and Sokolov defined a 1-parameter family of Poisson brackets on the space $\mathcal{W}/\partial\mathcal{W}$ of local functionals on an infinite-dimensional Poisson manifold \mathcal{M} . Such Poisson manifold is obtained, starting from an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$, via a Hamiltonian reduction, and the corresponding differential algebra \mathcal{W} of functions on \mathcal{M} , with its Poisson bracket on the space $\mathcal{W}/\partial\mathcal{W}$ of local functionals, is known as classical \mathcal{W} -algebra. In the same paper Drinfeld and Sokolov constructed an integrable hierarchy of bi-Hamiltonian equations associated to each classical \mathcal{W} -algebra, known as “generalized KdV hierarchy”. The Korteweg-de Vries (KdV) equation appears in the case of $\mathfrak{g} = \mathfrak{sl}_2$. For $\mathfrak{g} = \mathfrak{sl}_n$, the classical \mathcal{W} -algebra Poisson bracket coincides with the Adler-Gelfand-Dickey Poisson bracket [Adl79, GD87] on the space of local functionals on the set of ordinary differential operators of the form $\partial^n + u_1\partial^{n-2} + \dots + u_{n-1}$, and the corresponding integrable hierarchy is the so called n -th Gelfand-Dickey hierarchy (see [Dic97] for a review).

In a few words, the construction of [DS85] is as follows. Let \mathfrak{g} be a simple finite-dimensional Lie algebra with a non-degenerate symmetric invariant bilinear form κ , and let f be a principal nilpotent element in \mathfrak{g} , which we include in an \mathfrak{sl}_2 -triple $(f, h = 2x, e)$ in \mathfrak{g} . Then \mathfrak{g} decomposes as a direct sum of $\text{ad } x$ -eigenspaces $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. To define the Poisson manifold \mathcal{M} , consider first the space $\widetilde{\mathcal{M}}$ of first order differential operators of the form

$$L(z) = \partial_x + f + zs + q(x),$$

where s lies in the center of $\mathfrak{n}_+ = \bigoplus_{i>0} \mathfrak{g}_i$, $q(x)$ is a smooth map $S^1 \rightarrow \mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{n}_+$, and z is an indeterminate. On this space there is an action of the infinite-dimensional Lie group N , whose Lie algebra is the space of smooth maps $S^1 \rightarrow \mathfrak{n}_+$, by gauge transformations:

$$L^A(z) = e^{\text{ad } A} L(z),$$

for any smooth map $A : S^1 \rightarrow \mathfrak{n}_+$. The Poisson manifold \mathcal{M} is then obtained as the quotient of $\widetilde{\mathcal{M}}$ by the action of the gauge group: $\mathcal{M} = \widetilde{\mathcal{M}}/N$. As a differential algebra, the classical \mathcal{W} -algebra is therefore the space of functions on $\widetilde{\mathcal{M}}$ which are gauge invariant. The corresponding 1-parameter family of Poisson brackets on $\mathcal{W}/\partial\mathcal{W}$ is obtained as a reduction of the affine algebra Lie-Poisson bracket. An explicit formula for it is

$$\{f g, f h\}_{z,\rho} = \int \kappa \left(\frac{\delta h}{\delta q} \mid \left[L(z), \frac{\delta g}{\delta q} \right] \right),$$

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where $\frac{\delta g}{\delta q}$ denotes the variational derivative of the local functional $\int g \in \mathcal{W}/\partial\mathcal{W}$ (the index ρ will be explained in Section 3.2).

In order to construct an integrable hierarchy of bi-Hamiltonian equations for \mathcal{W} , one conjugates L to an operator of the form

$$L_0(z) = e^{\text{ad } U(z)} L(z) = \partial_x + f + zs + h(z),$$

where $U(z)$ is a smooth function on S^1 with values in $\mathfrak{n}_+ \oplus \mathfrak{g}[[z^{-1}]]z^{-1}$, and $h(z)$ is a smooth function on S^1 with values in $\mathfrak{h} \cap \mathfrak{g}[[z^{-1}]]$, where $\mathfrak{h} = \text{Ker ad}(f + zs)$ (it is an abelian subalgebra of $\mathfrak{g}((z^{-1}))$). Then, for any element $a(z) \in \mathfrak{h}$ we obtain an infinite sequence of Hamiltonian functionals in involution defined by ($n \in \mathbb{Z}_+$):

$$\int \mathcal{H}_n = \int \text{Res}_z z^{n-1} \kappa(a(z)|h(z)) \in \mathcal{W}/\partial\mathcal{W}.$$

The corresponding generalized KdV hierarchy of Hamiltonian equations is $\frac{dp}{dt_n} = \{\int \mathcal{H}_n, \int p\}_{0,\rho}$, $n \in \mathbb{Z}_+$.

Since the original paper of Drinfeld and Sokolov, the construction of the classical \mathcal{W} -algebras has been generalized by many authors to the case when $f \in \mathfrak{g}$ is an arbitrary nilpotent element. In the framework of Poisson vertex algebras, they have been constructed in [DSK06]. In [dGHM92, BdGHM93, FGMS95, FGMS96] they constructed the corresponding generalized KdV hierarchies, starting with a Heisenberg subalgebra $\mathcal{H} \subset \mathfrak{g}((z^{-1}))$. In this approach, they cover all classical \mathcal{W} -algebras associated to nilpotent elements $f \in \mathfrak{g}$, for which there exists a graded semisimple element of the form $f + zs \in \mathcal{H}$ (the existence of such a graded semisimple element is also studied, in the regular, or “type I”, case, in [FHM92, DF95], using results in [KP85], and, for \mathfrak{g} of type A_n , in [FGMS95, FGMS96]).

In [BDSK09] the theory of Hamiltonian equations and integrable bi-Hamiltonian hierarchies has been naturally related to the theory of Poisson vertex algebras.

Recall that a *Poisson vertex algebra* (PVA) is a differential algebra \mathcal{V} , with a derivation ∂ , endowed with a λ -bracket $\{\cdot_\lambda \cdot\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathbb{F}[\lambda] \otimes \mathcal{V}$ satisfying sesquilinearity (1.4), left and right Leibniz rules (1.5)-(1.6), skew-symmetry (1.7), and Jacobi identity (1.8), displayed in Section 1. Given a PVA structure on an algebra \mathcal{V} of smooth functions $u : S^1 \rightarrow \mathbb{R}$, or, in a more algebraic context, on an algebra of differential polynomials \mathcal{V} over a field \mathbb{F} of characteristics zero, and a local Hamiltonian functional $\int h \in \mathcal{V}/\partial\mathcal{V}$, the corresponding *Hamiltonian equation* is

$$\frac{du}{dt} = \{h_\lambda u\}|_{\lambda=0}. \quad (0.1)$$

An *integral of motion* for such evolution equation is a local functional $\int g \in \mathcal{V}/\partial\mathcal{V}$ such that

$$\{\int h, \int g\} := \int \{h_\lambda g\}|_{\lambda=0} = 0.$$

Equation (0.1) is said to be *integrable* if there exists an infinite sequence $\int h_0 = \int h, \int h_1, \int h_2, \dots$ of linearly independent integrals of motion in involution: $\{\int h_m, \int h_n\} = 0$, for all $m, n \in \mathbb{Z}_+$.

The main tool to construct an infinite hierarchy of Hamiltonian equations is the so called Lenard-Magri scheme (see [Mag78]). This scheme can be applied to a bi-Hamiltonian equation, that is an evolution equation which can be written in two compatible Hamiltonian forms:

$$\frac{du}{dt} = \{h_{0\lambda} u\}_H|_{\lambda=0} = \{h_{1\lambda} u\}_K|_{\lambda=0}, \quad (0.2)$$

where $\{\cdot_\lambda \cdot\}_H$ and $\{\cdot_\lambda \cdot\}_K$ are compatible λ -brackets, in the sense that any their linear combination defines a PVA structure on \mathcal{V} . In this case, under some additional conditions, one can solve the recurrence equation

$$\{h_{n\lambda} u\}_H = \{h_{n+1\lambda} u\}_K, \quad n \in \mathbb{Z}_+.$$

Then, according to the Lenard-Magri scheme, the local functionals $\int h_n$, $n \in \mathbb{Z}_+$, are in involution, so that equation (0.2) is integrable, provided that the $\int h_n$ ’s are linearly independent.

The main aim of the present paper is to derive the Drinfeld-Sokolov construction of classical \mathcal{W} -algebras and generalized KdV hierarchies, as well as the generalizations mentioned above, within the context of Poisson vertex algebras.

In fact, it appears clear from the results in Section 3, that Poisson vertex algebras provide the most natural framework to describe classical \mathcal{W} -algebras and the corresponding generalized Drinfeld-Sokolov Hamiltonian reduction. In particular, Theorem 3.8 (and the following Remark 3.9) shows that the action of the gauge group N on the phase space $\widetilde{\mathcal{M}}$ coincides with an action of a “Lie conformal group” on the space $\widetilde{\mathcal{W}}$ of functions on $\widetilde{\mathcal{M}}$, obtained by exponentiating the natural Lie conformal algebra action of $\mathbb{F}[\partial]\mathfrak{n}$ on $\widetilde{\mathcal{W}}$, where \mathfrak{n} is a certain subalgebra of \mathfrak{n}_+ .

The paper is organized as follows. In Section 1 we review, following [BDSK09], the basic definitions and notations of Poisson vertex algebra theory and its application to the theory of integrable bi-Hamiltonian equations.

In Section 2, following the original ideas of Drinfeld and Sokolov, we show how to apply the Lenard-Magri scheme of integrability for the affine Poisson vertex algebra $\mathcal{V}(\mathfrak{g})$ (defined in Example 1.4), where \mathfrak{g} is a reductive Lie algebra. This is known as the homogeneous case, as it corresponds to the choice $f = 0$. The main result here is Corollary 2.9, which provides an integrable hierarchy of Hamiltonian equations in $\mathcal{V}(\mathfrak{g})$, associated to a semisimple element $s \in \mathfrak{g}$ and an element $a \in Z(\text{Ker}(\text{ad } s)) \setminus Z(\mathfrak{g})$.

Section 3 is the heart of the paper. We first define the action of the Lie conformal algebra $\mathbb{F}[\partial]\mathfrak{n}$ on a suitable differential subalgebra $\mathcal{V}(\mathfrak{p})$ of $\mathcal{V}(\mathfrak{g})$. The classical \mathcal{W} -algebra is then defined as $\mathcal{V}(\mathfrak{p})^{\mathbb{F}[\partial]\mathfrak{n}}$, that is the subspace of $\mathbb{F}[\partial]\mathfrak{n}$ -invariants in $\mathcal{V}(\mathfrak{p})$. As discussed above, in Section 3.3, we show how this Lie conformal algebra action is related to the action of the gauge group N on the phase space $\widetilde{\mathcal{M}}$, and we then prove that our definition of classical \mathcal{W} -algebra is equivalent to the original definition of Drinfeld and Sokolov. Then, in Section 3.4, we use this correspondence to prove that the classical \mathcal{W} -algebra is an algebra of differential polynomials in $r = \dim(\text{Ker } \text{ad } f)$ variables, and to give an explicit set of generators for it.

Finally, in Section 4, following the ideas of Drinfeld and Sokolov, we apply the Lenard-Magri scheme of integrability to derive integrable hierarchies for classical \mathcal{W} -algebras. The main result here is Theorem 4.18, where we construct an integrable hierarchy of bi-Hamiltonian equations associated to a nilpotent element $f \in \mathfrak{g}$, a homogeneous element $s \in \mathfrak{g}$ such that $[s, \mathfrak{n}] = 0$ and $f + zs \in \mathfrak{g}((z^{-1}))$ is semisimple, and an element $a(z) \in Z(\text{Ker } \text{ad}(f + zs)) \setminus Z(\mathfrak{g}((z^{-1})))$. In Section 4.10 we discuss, in the case of \mathfrak{gl}_n , for which nilpotent elements f Theorem 4.18 can be applied, obtaining the same restrictions as in [FGMS95].

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1. POISSON VERTEX ALGEBRAS AND HAMILTONIAN EQUATIONS

In this section we review the connection between Poisson vertex algebras and the theory of Hamiltonian equations as laid down in [BDSK09]. It is shown that Poisson vertex algebras provide a convenient framework for systems of Hamiltonian equations. As the main application we explain how to establish integrability of such partial differential equations using the Lenard-Magri scheme.

1.1. Algebras of differential polynomials. By a *differential algebra* we mean a unital commutative associative algebra \mathcal{V} over a field \mathbb{F} of characteristic 0, with a derivation ∂ , that is an \mathbb{F} -linear map from \mathcal{V} to itself such that, for $a, b \in \mathcal{V}$

$$\partial(ab) = \partial(a)b + a\partial(b).$$

In particular $\partial 1 = 0$.

The most important examples we are interested in are the *algebras of differential polynomials* in the variables u_1, \dots, u_ℓ :

$$\mathcal{V} = \mathbb{F}[u_i^{(n)} \mid i \in I = \{1, \dots, \ell\}, n \in \mathbb{Z}_+],$$

where ∂ is the derivation defined by $\partial(u_i^{(n)}) = u_i^{(n+1)}$, $i \in I, n \in \mathbb{Z}_+$. Note that we have in \mathcal{V} the following commutation relations:

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}}, \quad (1.1)$$

where the RHS is considered to be zero if $n = 0$, which can be written equivalently in terms of generating series, as follows

$$\sum_{n \in \mathbb{Z}_+} z^n \frac{\partial(\partial g)}{\partial u_i^{(n)}} = (\partial + z) \sum_{n \in \mathbb{Z}_+} z^n \frac{\partial g}{\partial u_i^{(n)}}. \quad (1.2)$$

We say that $f \in \mathcal{V} \setminus \mathbb{F}$ has *differential order* $m \in \mathbb{Z}_+$ if $\frac{\partial f}{\partial u_i^{(m)}} \neq 0$ for some $i \in I$, and $\frac{\partial f}{\partial u_j^{(n)}} = 0$ for all $j \in I$ and $n > m$.

The *variational derivative* of $f \in \mathcal{V}$ with respect to u_i is, by definition,

$$\frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}.$$

It is immediate to check, using (1.2), that $\frac{\delta}{\delta u_i} \circ \partial = 0$ for every i . In the algebra \mathcal{V} of differential polynomials the converse is true too: if $\frac{\delta f}{\delta u_i} = 0$ for every $i = 1, \dots, \ell$, then necessarily f lies in $\partial\mathcal{V} \oplus \mathbb{F}$ (see e.g. [BDSK09]). Letting $U = \bigoplus_{i \in I} \mathbb{F}u_i$ be the generating space of \mathcal{V} , we define the *variational derivative* of $f \in \mathcal{V}$ as

$$\frac{\delta f}{\delta u} = \sum_{i \in I} u_i \otimes \frac{\delta f}{\delta u_i} \in U \otimes \mathcal{V}. \quad (1.3)$$

1.2. Poisson vertex algebras.

Definition 1.1. Let \mathcal{V} be a differential algebra. A λ -bracket on \mathcal{V} is an \mathbb{F} -linear map $\mathcal{V} \otimes \mathcal{V} \rightarrow \mathbb{F}[\lambda] \otimes \mathcal{V}$, denoted by $f \otimes g \rightarrow \{f_\lambda g\}$, satisfying *sesquilinearity* ($f, g \in \mathcal{V}$):

$$\{\partial f_\lambda g\} = -\lambda \{f_\lambda g\}, \quad \{f_\lambda \partial g\} = (\lambda + \partial) \{f_\lambda g\}, \quad (1.4)$$

and the *left and right Leibniz rules* ($f, g, h \in \mathcal{V}$):

$$\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g, \quad (1.5)$$

$$\{fh_\lambda g\} = \{f_{\lambda+\partial}g\}_{\rightarrow}h + \{h_{\lambda+\partial}g\}_{\rightarrow}f, \quad (1.6)$$

where we use the following notation: if $\{f_\lambda g\} = \sum_{n \in \mathbb{Z}_+} \lambda^n c_n$, then $\{f_{\lambda+\partial}g\}_{\rightarrow}h = \sum_{n \in \mathbb{Z}_+} c_n(\lambda + \partial)^n h$. We say that the λ -bracket is *skew-symmetric* if

$$\{g_\lambda f\} = -\{f_{-\lambda-\partial}g\}, \quad (1.7)$$

where, now, $\{f_{-\lambda-\partial}g\} = \sum_{n \in \mathbb{Z}_+} (-\lambda - \partial)^n c_n$ (if there is no arrow we move ∂ to the left).

Definition 1.2. A *Poisson vertex algebra* (PVA) is a differential algebra \mathcal{V} endowed with a λ -bracket which is skew-symmetric and satisfies the following *Jacobi identity* in $\mathcal{V}[\lambda, \mu]$ ($f, g, h \in \mathcal{V}$):

$$\{f_\lambda \{g_\mu h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\} + \{g_\mu \{f_\lambda h\}\}. \quad (1.8)$$

In this paper we consider PVA structures on an algebra of differential polynomials \mathcal{V} in the variables $\{u_i\}_{i \in I}$. In this case, thanks to sesquilinearity and Leibniz rules, the λ -brackets $\{u_i \lambda u_j\}$, $i, j \in I$, completely determines the λ -bracket on the whole algebra \mathcal{V} .

Theorem 1.3 ([BDSK09, Theorem 1.15]). *Let \mathcal{V} be an algebra of differential polynomials in the variables $\{u_i\}_{i \in I}$, and let $H_{ij}(\lambda) \in \mathbb{F}[\lambda] \otimes \mathcal{V}$, $i, j \in I$.*

(a) *The Master Formula*

$$\{f_\lambda g\} = \sum_{\substack{i, j \in I \\ m, n \in \mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} (\lambda + \partial)^n H_{ji}(\lambda + \partial) (-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}} \quad (1.9)$$

defines the λ -bracket on \mathcal{V} with given $\{u_i \lambda u_j\} = H_{ji}(\lambda)$, $i, j \in I$.

(b) *The λ -bracket (1.9) on \mathcal{V} satisfies the skew-symmetry condition (1.7) provided that the same holds on generators ($i, j \in I$):*

$$\{u_i \lambda u_j\} = -\{u_j_{-\lambda-\partial} u_i\}, \quad (1.10)$$

(c) *Assuming that the skew-symmetry condition (1.10) holds, the λ -bracket (1.9) satisfies the Jacobi identity (1.8), thus making \mathcal{V} a PVA, provided that the Jacobi identity holds on any triple of generators ($i, j, k \in I$):*

$$\{u_i \lambda \{u_j \mu u_k\}\} = \{\{u_i \lambda u_j\}_{\lambda+\mu} u_k\} + \{u_j \mu \{u_i \lambda u_k\}\}.$$

Example 1.4. Let \mathfrak{g} be a Lie algebra over \mathbb{F} with a symmetric invariant bilinear form κ , and let s be an element of \mathfrak{g} . The *affine PVA* $\mathcal{V}(\mathfrak{g}, \kappa, s)$, associated to the triple $(\mathfrak{g}, \kappa, s)$, is the algebra of differential polynomials $\mathcal{V} = S(\mathbb{F}[\partial]\mathfrak{g})$ (where $\mathbb{F}[\partial]\mathfrak{g}$ is the free $\mathbb{F}[\partial]$ -module generated by \mathfrak{g} and $S(R)$ denotes the symmetric algebra over the \mathbb{F} -vector space R) together with the λ -bracket given by

$$\{a_\lambda b\} = [a, b] + \kappa(s \mid [a, b]) + \kappa(a \mid b)\lambda \quad \text{for } a, b \in \mathfrak{g}, \quad (1.11)$$

and extended to \mathcal{V} by sesquilinearity and the left and right Leibniz rules.

In Section 3 we will define classical \mathcal{W} -algebras in terms of representations of Lie conformal algebras. Let us recall here the definitions [Kac98].

Definition 1.5. (a) A *Lie conformal algebra* is an $\mathbb{F}[\partial]$ -module R with an \mathbb{F} -linear map $\{\cdot \lambda \cdot\} : R \otimes R \rightarrow \mathbb{F}[\lambda] \otimes R$ satisfying (1.4), (1.7) and (1.8).

- (b) A *representation* of a Lie conformal algebra R on an $\mathbb{F}[\partial]$ -module \mathcal{V} is a λ -action $R \otimes \mathcal{V} \rightarrow \mathbb{F}[\lambda] \otimes \mathcal{V}$, denoted $a \otimes g \mapsto a \otimes_\lambda g$, satisfying sesquilinearity, $(\partial a) \otimes_\lambda g = -\lambda a \otimes_\lambda g$, $a \otimes_\lambda (\partial g) = (\lambda + \partial) a \otimes_\lambda g$, and Jacobi identity $a \otimes_\lambda (b \otimes_\mu g) - b \otimes_\mu (a \otimes_\lambda g) = \{a_\lambda b\} \otimes_{\lambda+\mu} g$ ($a, b \in R, g \in \mathcal{V}$).
- (c) If, moreover, \mathcal{V} is a differential algebra, we say the the action of R on \mathcal{V} is by *conformal derivations* if $a \otimes_\lambda (gh) = (a \otimes_\lambda g)h + (a \otimes_\lambda h)g$.

1.3. Hamiltonian structures and Hamiltonian equations. By Theorem 1.3(a), if \mathcal{V} is an algebra of differential polynomials in the variables $\{u_i\}_{i \in I}$, there is a bijective correspondence between $\ell \times \ell$ -matrices $H(\lambda) = (H_{ij}(\lambda))_{i,j \in I} \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$ and the λ -brackets $\{\cdot_\lambda \cdot\}_H$ on \mathcal{V} .

Let $U = \bigoplus_{i \in I} \mathbb{F} u_i$ be the generating space of \mathcal{V} , and let $\{\chi^i\}_{i \in I}$ be the dual basis of U^* . We have a natural identification

$$\text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda] \hookrightarrow \text{Hom}(U \otimes \mathcal{V}, U^* \otimes \mathcal{V}), \quad (1.12)$$

associating to the matrix $H = (H_{ij}(\lambda))_{i,j \in I}$ the linear map $U \otimes \mathcal{V} \rightarrow U^* \otimes \mathcal{V}$ given by

$$a \otimes f \mapsto \sum_{i,j \in I} \chi^j(a) \chi^i \otimes H_{ij}(\partial) f.$$

By an abuse of notation, from now on we will denote by the same letter an element $H \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$ or the corresponding linear map $H : U \otimes \mathcal{V} \rightarrow U^* \otimes \mathcal{V}$.

Definition 1.6. A *Hamiltonian structure* on \mathcal{V} is a matrix $H \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$ such that the corresponding λ -bracket $\{\cdot_\lambda \cdot\}_H$ defines a PVA structure on \mathcal{V} .

Example 1.7. Consider the affine PVA $\mathcal{V}(\mathfrak{g}, \kappa, s)$ defined in Example 1.4. Let $\{u_i\}_{i \in I}$ be a basis of \mathfrak{g} and let $\{\chi^i\}_{i \in I} \subset \mathfrak{g}^*$ be its dual basis. The corresponding Hamiltonian structure $H = (H_{ij}(\lambda)) \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$ to the λ -bracket defined in (1.11) is given by

$$H_{ij}(\lambda) = \{u_j \lambda u_i\} = [u_j, u_i] + \kappa(s \mid [u_j, u_i]) + \kappa(u_i \mid u_j) \lambda.$$

Via the identifications (1.12), H corresponds to the linear map $\mathfrak{g} \otimes \mathcal{V} \rightarrow \mathfrak{g}^* \otimes \mathcal{V}$ given by

$$H(a \otimes f) = \sum_{i \in I} \chi^i \otimes [a, u_i] f + \kappa([s, a] \mid \cdot) \otimes f + \kappa(a \mid \cdot) \otimes \partial f. \quad (1.13)$$

In the special case when κ is non-degenerate, we can identify $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$ via the isomorphism $\kappa(a \mid \cdot) \mapsto a$. Let $\{u^i\}_{i \in I} \subset \mathfrak{g}$ be the dual basis of \mathfrak{g} with respect to κ : $\kappa(u^i \mid u_j) = \delta_{ij}$, $i, j \in I$. Then, the map (1.13) is identified with the linear map $\mathfrak{g} \otimes \mathcal{V} \rightarrow \mathfrak{g} \otimes \mathcal{V}$ given by

$$H(a \otimes f) = \sum_{i \in I} [u^i, a] \otimes u_i f + [s, a] \otimes f + a \otimes \partial f. \quad (1.14)$$

The relation between PVAs and systems of Hamiltonian equations associated to a Hamiltonian structure is based on the following simple observation.

Proposition 1.8. Let \mathcal{V} be a PVA. The 0-th product on \mathcal{V} induces a well defined Lie algebra bracket on the quotient space $\mathcal{V}/\partial\mathcal{V}$:

$$\{\int f, \int g\} = \int \{f_\lambda g\}|_{\lambda=0}, \quad (1.15)$$

where $\int : \mathcal{V} \rightarrow \mathcal{V}/\partial\mathcal{V}$ is the canonical quotient map. Moreover, we have a well defined Lie algebra action of $\mathcal{V}/\partial\mathcal{V}$ on \mathcal{V} by derivations of the commutative associative product on \mathcal{V} , commuting with ∂ , given by

$$\{\int f, g\} = \{f_\lambda g\}|_{\lambda=0}.$$

In the special case when \mathcal{V} is an algebra of differential polynomials in ℓ variables $\{u_i\}_{i \in I}$ and the PVA λ -bracket on \mathcal{V} is associated to the Hamiltonian structure $H \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$, the Lie bracket (1.15) on $\mathcal{V}/\partial\mathcal{V}$ takes the form (see (1.9)):

$$\{\int f, \int g\} = \sum_{i,j \in I} \int \frac{\delta g}{\delta u_j} H_{ji}(\partial) \frac{\delta f}{\delta u_i}. \quad (1.16)$$

Definition 1.9. Let \mathcal{V} be an algebra of differential polynomials with a Hamiltonian structure H .

- (a) Elements of $\mathcal{V}/\partial\mathcal{V}$ are called *local functionals*.
- (b) Given a local functional $\int h \in \mathcal{V}/\partial\mathcal{V}$, the corresponding *Hamiltonian equation* is

$$\frac{du}{dt} = \{\int h, u\}_H \quad \left(\text{equivalently, } \frac{du_i}{dt} = \sum_{j \in I} H_{ij}(\partial) \frac{\delta h}{\delta u_j}, i \in I \right). \quad (1.17)$$

- (c) A local functional $\int f \in \mathcal{V}/\partial\mathcal{V}$ is called an *integral of motion* of equation (1.17) if $\frac{df}{dt} = 0 \pmod{\partial\mathcal{V}}$ in virtue of (1.17), or, equivalently, if $\int h$ and $\int f$ are *in involution*:

$$\{\int h, \int f\}_H = 0.$$

Namely, $\int f$ lies in the centralizer of $\int h$ in the Lie algebra $\mathcal{V}/\partial\mathcal{V}$ with Lie bracket (1.16).

- (d) Equation (1.17) is called *integrable* if there exists an infinite sequence $\int f_0 = \int h, \int f_1, \int f_2, \dots$, of linearly independent integrals of motion in involution. The corresponding *integrable hierarchy of Hamiltonian equations* is

$$\frac{du}{dt_n} = \{\int f_n, u\}_H, \quad n \in \mathbb{Z}_+ \quad \left(\text{equivalently, } \frac{du_i}{dt_n} = \sum_{j \in I} H_{ij}(\partial) \frac{\delta f_n}{\delta u_j}, \quad n \in \mathbb{Z}_+, i \in I \right). \quad (1.18)$$

1.4. Bi-Hamiltonian structures and integrability of Hamiltonian equations.

Definition 1.10. A *bi-Hamiltonian structure* (H, K) is a pair of Hamiltonian structures on an algebra of differential polynomials \mathcal{V} , which are compatible, in the sense that any \mathbb{F} -linear combination of them is a Hamiltonian structure.

Example 1.11. Consider Example 1.7, with non-degenerate κ . We identify $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$ via $\kappa(a \mid \cdot) \mapsto a$ and we describe a Hamiltonian structure on $\mathcal{V} = \mathcal{V}(\mathfrak{g}, \kappa, s)$ as an element of $\text{End}(\mathfrak{g} \otimes \mathcal{V})$ via the identification (1.12). Then, the maps $H, K : \mathfrak{g} \otimes \mathcal{V} \rightarrow \mathfrak{g} \otimes \mathcal{V}$ given by

$$H(a \otimes f) = \sum_{i \in I} [u^i, a] \otimes u_i f + a \otimes \partial f, \quad K(a \otimes f) = [a, s] \otimes f, \quad (1.19)$$

form a bi-Hamiltonian structure on \mathcal{V} . Comparing (1.14) and (1.19), we get that the Hamiltonian structure of Example 1.7 is equal to $H - K$.

Let \mathcal{V} be an algebra of differential polynomials, with the generating space $U \subset \mathcal{V}$, and we let (H, K) be a bi-Hamiltonian structure on \mathcal{V} . According to the *Lenard-Magri scheme of integrability* [Mag78] (see also [BDSK09]), in order to obtain an integrable hierarchy of Hamiltonian equations, one needs to find a sequence of local functionals $\{\int f_n\}_{n \in \mathbb{Z}_+}$ spanning an infinite-dimensional subspace of $\mathcal{V}/\partial\mathcal{V}$, such that their variational derivatives $F_n = \frac{\delta f_n}{\delta u} \in U \otimes \mathcal{V}$, $n \in \mathbb{Z}_+$, satisfy

$$K(F_0) = 0, \quad H(F_n) = K(F_{n+1}) \in U^* \otimes \mathcal{V} \quad \text{for every } n \in \mathbb{Z}_+. \quad (1.20)$$

If this is the case the elements $\int f_n$, $n \in \mathbb{Z}_+$, form an infinite sequence of local functionals in involution: $\{\int f_m, \int f_n\}_H = \{\int f_m, \int f_n\}_K = 0$, for all $m, n \in \mathbb{Z}_+$. Hence, we get the corresponding integrable hierarchy of Hamiltonian equations (1.18).

We note that, using the generating series $F(z) = \sum_{n \in \mathbb{Z}_+} z^{-n} \frac{\delta f_n}{\delta u} \in (U \otimes \mathcal{V})[[z^{-1}]]$, we can rewrite the Lenard-Magri recursion (1.20) as:

$$K(F_0) = 0, \quad (H - zK)F(z) = 0. \quad (1.21)$$

2. DRINFELD-SOKOLOV HIERARCHIES IN THE HOMOGENEOUS CASE

As the first application of the Lenard-Magri scheme, we construct integrable hierarchies of Hamiltonian equations (1.17) for the bi-Hamiltonian structure provided by Example 1.11. This is referred to as the homogeneous Drinfeld-Sokolov hierarchy. The non-homogeneous case will be treated in the next sections, after giving the definition of classical \mathcal{W} -algebras.

Let \mathfrak{g} be a finite-dimensional Lie algebra with a non-degenerate symmetric invariant bilinear form κ , let s be an element of \mathfrak{g} , and let $\mathcal{V} = S(\mathbb{F}[\partial]\mathfrak{g})$. Recall from Example 1.11 that we have a bi-Hamiltonian structure $H, K : \mathfrak{g} \otimes \mathcal{V} \rightarrow \mathfrak{g} \otimes \mathcal{V}$ on \mathcal{V} , given by (1.19) (as before, we are using the identification (1.12) for Hamiltonian structures, and the isomorphism $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$ associated to the bilinear form κ).

We endow the space $\mathfrak{g} \otimes \mathcal{V}$ with a Lie algebra structure letting, for $a, b \in \mathfrak{g}$ and $f, g \in \mathcal{V}$,

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg \in \mathfrak{g} \otimes \mathcal{V}.$$

We extend the bilinear form κ of \mathfrak{g} to a bilinear map $\kappa : \mathfrak{g} \otimes \mathcal{V} \times \mathfrak{g} \otimes \mathcal{V} \rightarrow \mathcal{V}$, given by $(a, b \in \mathfrak{g}, f, g \in \mathcal{V})$:

$$\kappa(a \otimes f \mid b \otimes g) = \kappa(a \mid b)fg \in \mathcal{V}. \quad (2.1)$$

Clearly, this bilinear map κ is symmetric invariant and non-degenerate. We extend $\partial \in \text{Der}(\mathcal{V})$ to a derivation of the Lie algebra $\mathfrak{g} \otimes \mathcal{V}$ given by

$$\partial(a \otimes f) = a \otimes \partial f,$$

for any $a \in \mathfrak{g}$ and $f \in \mathcal{V}$. Thus we get the semidirect product Lie algebra $\mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V})$, where $[\partial, a \otimes f] = \partial(a \otimes f) = a \otimes \partial f$ for $a \in \mathfrak{g}$ and $f \in \mathcal{V}$.

We set $\tilde{\mathfrak{g}} = (\mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V}))((z^{-1}))$, the space of Laurent series in z^{-1} with coefficients in $\mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V})$, endowed with the Lie algebra structure induced by that on $\mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V})$. We note that $(\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]] \subset \tilde{\mathfrak{g}}$ is a Lie subalgebra.

Proposition 2.1. *Let $L(z) = \partial + u + zs \otimes 1 \in \tilde{\mathfrak{g}}$, where $u = \sum_{i \in I} u^i \otimes u_i \in \mathfrak{g} \otimes \mathcal{V}$. Then*

$$(H - zK)(a \otimes f) = [L(z), a \otimes f],$$

for any $a \in \mathfrak{g}$ and $f \in \mathcal{V}$.

Proof. It follows immediately by (1.19) and the definition of the Lie bracket on $\tilde{\mathfrak{g}}$. \square

Recall from Section 1.4 that, to construct an integrable hierarchy of Hamiltonian equations, we need to find $\int f(z) \in (\mathcal{V}/\partial\mathcal{V})[[z^{-1}]]$ such that $F(z) = \frac{\delta f(z)}{\delta u} \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]$ is solution of equation (1.21). Hence, using Proposition 2.1, we conclude that the Lenard-Magri scheme can be applied if there exists $F(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]$ satisfying the following three conditions:

$$(C1) \quad [s \otimes 1, F_0] = 0,$$

$$(C2) \quad [L(z), F(z)] = 0.$$

$$(C3) \quad F(z) = \frac{\delta f(z)}{\delta u}, \text{ for some } \int f(z) \in (\mathcal{V}/\partial\mathcal{V})[[z^{-1}]].$$

The solution of the above problem will be achieved in Propositions 2.3 and 2.4 below (which are due to Drinfeld and Sokolov [DS85]), under the assumption that $s \in \mathfrak{g}$ is a semisimple element: in Proposition 2.3 we find an element $F(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]$ satisfying conditions (C1) and (C2), and in Proposition 2.4 we show that this element satisfies condition (C3).

Before stating the results we need to introduce some notation. For $U(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$, we have a well-defined Lie algebra automorphism

$$e^{\text{ad } U(z)} : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}.$$

By the Baker-Campbell-Hausdorff formula [Ser92], automorphisms of this type form a group. Fix a semisimple element $s \in \mathfrak{g}$, and denote $\mathfrak{h} = \text{Ker}(\text{ad } s) \subset \mathfrak{g}$ (it is clearly a subalgebra). By invariance of the bilinear form κ we have that $\mathfrak{h}^\perp = \text{Im}(\text{ad } s)$, and that $\mathfrak{g} = \text{Ker}(\text{ad } s) \oplus \text{Im}(\text{ad } s)$.

Remark 2.2. We can replace the assumption that $s \in \mathfrak{g}$ is semisimple by the assumption that \mathfrak{g} admits a vector space decomposition $\mathfrak{g} = \text{Ker}(\text{ad } s) \oplus \text{Im}(\text{ad } s)$. Clearly, if $s \in \mathfrak{g}$ is semisimple (that is, $\text{ad } s$ is diagonalizable in \mathfrak{g}), then automatically $\mathfrak{g} = \text{Ker}(\text{ad } s) \oplus \text{Im}(\text{ad } s)$. It is not hard to show that the converse is true if \mathfrak{g} is a reductive Lie algebra.

Proposition 2.3. (a) *There exist unique formal series $U(z) \in (\mathfrak{h}^\perp \otimes \mathcal{V})[[z^{-1}]]z^{-1}$ and $h(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]$ such that*

$$L_0(z) = e^{\text{ad } U(z)}(L(z)) = \partial + zs \otimes 1 + h(z). \quad (2.2)$$

(b) *An automorphism $e^{\text{ad } U(z)} \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$ solving (2.2) for some $h(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]$ is defined uniquely up to multiplication on the left by automorphisms of the form $e^{\text{ad } S(z)}$, where $S(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$.*

(c) *Let $a \in Z(\mathfrak{h})$ (the center of \mathfrak{h}), and let $U(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$, $h(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]$ solve equation (2.2). Then*

$$F(z) = e^{-\text{ad } U(z)}(a \otimes 1) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]] \quad (2.3)$$

is independent on the choice of $U(z)$ and it solves equations (C1) and (C2) above: $[s \otimes 1, F_0] = 0$ and $[L(z), F(z)] = 0$.

Proof. Writing $U(z) = \sum_{i \geq 1} U_i z^{-i}$ and $h(z) = \sum_{i \in \mathbb{Z}_+} h_i z^{-i}$, with $U_i, h_i \in \mathfrak{g} \otimes \mathcal{V}$, and equating coefficients of z^{-i} in both sides of (2.2), we find an equation of the form

$$h_i + [s \otimes 1, U_{i+1}] = A,$$

where $A \in \mathfrak{g} \otimes \mathcal{V}$ is expressed, inductively, in terms of U_1, U_2, \dots, U_i and h_0, h_1, \dots, h_{i-1} . For example, equating the constant term in (2.2) gives the relation $h_0 + [s \otimes 1, U_1] = u$, while, equating the coefficients of z^{-1} , gives the relation $h_1 + [s \otimes 1, U_2] = -U_1' + [U_1, u] + \frac{1}{2}[U_1, [U_1, s \otimes 1]]$. Decomposing $A = A_{\mathfrak{h}} + A_{\mathfrak{h}^\perp}$, according to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, we get $h_i = A_{\mathfrak{h}}$ and $[s \otimes 1, U_{i+1}] = A_{\mathfrak{h}^\perp}$, which uniquely defines $U_{i+1} \in \mathfrak{h}^\perp \otimes \mathcal{V}$, proving (a).

Let $U(z) \in (\mathfrak{h}^\perp \otimes \mathcal{V})[[z^{-1}]]z^{-1}$, $h(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]$ be the unique solution of (2.2) given by part (a), and let $\tilde{U}(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$, $\tilde{h}(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]$ be some other solution of (2.2): $e^{\text{ad } \tilde{U}(z)}(L(z)) =$

$\partial + zs \otimes 1 + \tilde{h}(z)$. By the observation before the statement of the proposition, there exists $S(z) = \sum_{i=1}^{\infty} S_i z^{-i} \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$ such that

$$e^{\text{ad } \tilde{U}(z)} = e^{\text{ad } S(z)} e^{\text{ad } U(z)}. \quad (2.4)$$

To conclude the proof of (b), we need to show that $S(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$. Applying equation (2.4) to $L(z)$ we get

$$\partial + zs \otimes 1 + \tilde{h}(z) = e^{\text{ad } S(z)} (\partial + zs \otimes 1 + h(z)). \quad (2.5)$$

Comparing the constant terms in z of both sides of (2.5), we get $\tilde{h}_0 = h_0 + [s \otimes 1, S_1]$, which clearly implies $\tilde{h}_0 = h_0$ and $S_1 \in \mathfrak{h} \otimes \mathcal{V}$. Assuming by induction that S_1, \dots, S_i lie in $\mathfrak{h} \otimes \mathcal{V}$, and comparing the coefficients of z^{-i} in both sides of (2.5) we get $[s \otimes 1, S_{i+1}] \in (\mathfrak{h}^\perp \cap \mathfrak{h}) \otimes \mathcal{V} = 0$, so that $S_{i+1} \in \mathfrak{h} \otimes \mathcal{V}$, as desired.

We are left to prove part (c). By part (b), $e^{-\text{ad } \tilde{U}(z)}(a \otimes 1) = e^{-\text{ad } U(z)} e^{-\text{ad } S(z)}(a \otimes 1) = F(z)$, since, by construction $S(z)$ lies in $(\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$ (hence, it commutes with $a \otimes 1 \in Z(\mathfrak{h}) \otimes \mathcal{V}$). Moreover, $F_0 = a \otimes 1$. Hence, it commutes with $[s \otimes 1]$, proving condition (C1). Finally, condition (C2) follows from the facts that $[L_0(z), a \otimes 1] = 0$ and $e^{-\text{ad } U(z)}$ is a Lie algebra automorphism of $\tilde{\mathfrak{g}}$. \square

Proposition 2.4. *Let $U(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$ and $h(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]$ be a solution of equation (2.2). Then the formal power series $F(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]$ defined in (2.3) (where $a \in Z(\mathfrak{h})$) satisfies condition (C3) above, namely $F(z) = \frac{\delta f(z)}{\delta u}$, where*

$$\int f(z) = \int \kappa(a \otimes 1 \mid h(z)) \in \mathcal{V}/\partial \mathcal{V}[[z^{-1}]]. \quad (2.6)$$

Before proving Proposition 2.4 we introduce some notation and we prove two preliminary lemmas. We extend the partial derivatives $\frac{\partial}{\partial u_i^{(m)}}$ to derivations of the Lie algebra $\mathfrak{g} \otimes \mathcal{V}$ in the obvious way: $\frac{\partial}{\partial u_i^{(m)}}(a \otimes f) = a \otimes \frac{\partial f}{\partial u_i^{(m)}}$. We also define the differential order of elements $F \in \mathfrak{g} \otimes \mathcal{V} \setminus \mathfrak{g} \otimes \mathbb{F}$ in the same way as before: F has differential order $m \in \mathbb{Z}_+$ if $\frac{\partial F}{\partial u_i^{(m)}} \neq 0$ for some $i \in I$, and $\frac{\partial F}{\partial u_j^{(n)}} = 0$ for all $j \in I$ and $n > m$.

Note that if $A \in \mathfrak{g} \otimes \mathcal{V}$ and $B \in \mathbb{F} \partial \ltimes (\mathfrak{g} \otimes \mathcal{V})$, we have $(\text{ad } A)B \in \mathfrak{g} \otimes \mathcal{V}$.

Lemma 2.5. *For $\alpha \in \mathbb{F}$, $A, U_1, \dots, U_k \in \mathfrak{g} \otimes \mathcal{V}$, with $k \geq 1$, we have*

$$\begin{aligned} \frac{\partial}{\partial u_i^{(m)}} \left(\text{ad}(U_1) \cdots \text{ad}(U_k) (\alpha \partial + A) \right) &= \sum_{h=1}^k \text{ad}(U_1) \cdots \text{ad} \left(\frac{\partial U_h}{\partial u_i^{(m)}} \right) \cdots \text{ad}(U_k) (\alpha \partial + A) \\ &\quad + \text{ad}(U_1) \cdots \text{ad}(U_k) \left(\frac{\partial A}{\partial u_i^{(m)}} \right) - \alpha \text{ad}(U_1) \cdots \text{ad}(U_{k-1}) \left(\frac{\partial U_k}{\partial u_i^{(m-1)}} \right). \end{aligned} \quad (2.7)$$

Proof. Equation (2.7) follows from the fact that $\frac{\partial}{\partial u_i^{(m)}}$ is a derivation of $\mathfrak{g} \otimes \mathcal{V}$ and by the commutation rule (1.1). \square

Lemma 2.6. *For $U, V \in \mathfrak{g} \otimes \mathcal{V}$ and $L \in \mathbb{F} \partial \ltimes (\mathfrak{g} \otimes \mathcal{V})$, we have*

$$\left[\sum_{h \in \mathbb{Z}_+} \frac{1}{(h+1)!} (\text{ad } U)^h(V), e^{\text{ad } U}(L) \right] = \sum_{h, k \in \mathbb{Z}_+} \frac{1}{(h+k+1)!} (\text{ad } U)^h (\text{ad } V) (\text{ad } U)^k(L) \quad (2.8)$$

Proof. Since $\text{ad } U$ is a derivation of the Lie bracket in $\mathbb{F} \partial \ltimes (\mathfrak{g} \otimes \mathcal{V})$, we have

$$\begin{aligned} \text{RHS}(2.8) &= \sum_{h, k \in \mathbb{Z}_+} \frac{(\text{ad } U)^h}{(h+k+1)!} [V, (\text{ad } U)^k(L)] = \sum_{h, k \in \mathbb{Z}_+} \sum_{l=0}^h \frac{\binom{h}{l}}{(h+k+1)!} [(\text{ad } U)^l(V), (\text{ad } U)^{h+k-l}(L)] \\ &= \sum_{m, n \in \mathbb{Z}_+} \frac{1}{(m+1)!} \frac{1}{n!} [(\text{ad } U)^m(V), (\text{ad } U)^n(L)] \sum_{h=m}^{m+n} \frac{\binom{h}{m}}{\binom{m+n+1}{m+1}}. \end{aligned}$$

The RHS above is the same as the LHS of (2.8) thanks to the simple combinatorial identity $\sum_{h=m}^{m+n} \binom{h}{m} = \binom{m+n+1}{m+1}$. \square

Proof of Proposition 2.4. We need to compute $\frac{\delta f(z)}{\delta u}$. By the definition (1.3) of the variational derivative and the definition (2.6) of $\int f(z)$, we have

$$\begin{aligned} \frac{\delta f(z)}{\delta u} &= \sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^m \kappa \left(a \otimes 1 \left| \frac{\partial h(z)}{\partial u_i^{(m)}} \right. \right) \\ &= \sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^m \kappa \left(a \otimes 1 \left| \frac{\partial}{\partial u_i^{(m)}} \left(e^{\text{ad } U(z)} (\partial + u + zs \otimes 1) - \partial - zs \otimes 1 \right) \right. \right) \\ &= a \otimes 1 + \sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^m \kappa \left(a \otimes 1 \left| \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial}{\partial u_i^{(m)}} (\text{ad } U(z))^k (\partial + u + zs \otimes 1) \right. \right). \end{aligned} \quad (2.9)$$

In the second identity we used the definition (2.2) of $h(z)$, and in the last identity we used the Taylor series expansion for the exponential $e^{\text{ad } U(z)}$. By Lemma 2.5, the last term in the RHS of (2.9) can be rewritten as

$$\begin{aligned} &\sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^m \kappa \left(a \otimes 1 \left| \sum_{k=1}^{\infty} \sum_{h=0}^{k-1} \frac{1}{k!} (\text{ad } U(z))^h \left(\text{ad } \frac{\partial U(z)}{\partial u_i^{(m)}} \right) (\text{ad } U(z))^{k-h-1} L(z) \right. \right. \\ &+ \sum_{k=1}^{\infty} \frac{1}{k!} (\text{ad } U(z))^k \frac{\partial}{\partial u_i^{(m)}} (u + zs \otimes 1) - \sum_{k \in \mathbb{Z}_+} \frac{1}{(k+1)!} (\text{ad } U(z))^k \frac{\partial U(z)}{\partial u_i^{(m-1)}} \Bigg) \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} (-\text{ad } U(z))^k (a \otimes 1) - \sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^m \kappa \left(a \otimes 1 \left| \sum_{k \in \mathbb{Z}_+} \frac{1}{(k+1)!} (\text{ad } U(z))^k \frac{\partial U(z)}{\partial u_i^{(m-1)}} \right. \right) \\ &+ \sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^m \kappa \left(a \otimes 1 \left| \sum_{h, k \in \mathbb{Z}_+} \frac{1}{(h+k+1)!} (\text{ad } U(z))^h \left(\text{ad } \frac{\partial U(z)}{\partial u_i^{(m)}} \right) (\text{ad } U(z))^k L(z) \right. \right). \end{aligned} \quad (2.10)$$

For the first term in the RHS we used the invariance of the bilinear map $\kappa : (\mathfrak{g} \otimes \mathcal{V}) \times (\mathfrak{g} \otimes \mathcal{V}) \rightarrow \mathcal{V}$. Combining the first term in the RHS of (2.9) and the first term in the RHS of (2.10), we get $e^{-\text{ad } U(z)}(a \otimes 1)$, which is the same as $F(z)$ by (2.3). Hence, in order to complete the proof of the proposition, we are left to show that the last two terms in the RHS of (2.10) cancel. Let

$$A_{i,m}(z) = \sum_{k \in \mathbb{Z}_+} \frac{1}{(k+1)!} (\text{ad } U(z))^k \frac{\partial U(z)}{\partial u_i^{(m)}}.$$

Using this notation, the second term of the RHS of (2.10) can be rewritten as

$$- \sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^m \kappa \left(a \otimes 1 \mid A_{i,m-1}(z) \right), \quad (2.11)$$

while, by Lemma 2.6, the third term of the RHS of (2.10) is

$$\sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^m \kappa \left(a \otimes 1 \mid [A_{i,m}(z), e^{\text{ad } U(z)} L(z)] \right).$$

By equation (2.2), the invariance of the bilinear map κ and the assumption that a lies in the center of \mathfrak{h} , the above expression is equal to

$$\sum_{i \in I, m \in \mathbb{Z}_+} u_i \otimes (-\partial)^{m+1} \kappa \left(a \otimes 1 \mid A_{i,m}(z) \right),$$

which, combined with (2.11), gives zero. \square

Remark 2.7. Consider the usual polynomial grading of the algebra of differential polynomials $\mathcal{V} = S(\mathbb{F}[\partial]\mathfrak{g})$. We can compute the part of $\int f(z) \in (\mathcal{V}/\partial\mathcal{V})[[z^{-1}]]$ of degree less or equal than 2, using equations (2.2) and (2.6). For $n \in \mathbb{Z}_+$, we denote $U(z)(n)$, $h(z)(n)$ and $\int f(z)(n)$ the homogeneous components of degree n in $U(z)$, $h(z)$ and $\int f(z)$ respectively. Using equation (2.2) and the fact that $\mathfrak{h} \cap \mathfrak{h}^\perp = 0$, it is easy to show, inductively on the negative powers of z , that $U(z)(0) = 0$ and $h(z)(0) = 0$, so that $\int f(z)(0) = 0$. Similarly, equating the homogeneous components of degree 1 in equation (2.2), we get $h(z)(1) = \pi_{\mathfrak{h}} u$, where $\pi_{\mathfrak{h}} : \mathfrak{g} \otimes \mathcal{V} \rightarrow \mathfrak{h} \otimes \mathcal{V}$ is the canonical quotient map (with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$). Hence, $\int f(z)(1) = \int \kappa(a \otimes 1 \mid u)$. Moreover, $U(z)(1)$ solves the equation

$$[zs \otimes 1, U(z)(1)] = \pi_{\mathfrak{h}^\perp} u - U'(z)(1).$$

More explicitly, the coefficient of z^{-n} in $U(z)(1)$ is given by $(\text{ad } s)^{-n}(-\partial)^{n-1}\pi_{\mathfrak{h}^\perp}u$, where $\text{ad } s$ is considered as an invertible endomorphism of \mathfrak{h}^\perp . Finally, equating the homogeneous components of degree 2 in (2.2), we get $h(z)(2) = \frac{1}{2}\pi_{\mathfrak{h}}[U(z)(1), u]$. Hence,

$$\int f(z)(2) = \frac{1}{2} \sum_{n=1}^{\infty} z^{-n} \int \kappa(a \otimes 1 \mid [(\text{ad } s)^{-n}(-\partial)^{n-1}\pi_{\mathfrak{h}^\perp}u, u]).$$

Remark 2.8. Let $U(z), h(z)$ and $\tilde{U}(z), \tilde{h}(z)$ be two solutions of (2.2). Recall by Proposition 2.3(b) that $e^{\text{ad } \tilde{U}(z)} = e^{\text{ad } S(z)}e^{\text{ad } U(z)}$ for some $S(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$. By Proposition 2.3(c), $F(z) = e^{-\text{ad } U(z)}(a \otimes 1) = e^{-\text{ad } \tilde{U}(z)}(a \otimes 1)$. Hence, by Proposition 2.4, $f(z) = \int \kappa(a \otimes 1 \mid h(z))$ and $\tilde{f}(z) = \int \kappa(a \otimes 1 \mid \tilde{h}(z))$ differ by a total derivative. In particular, if \mathfrak{h} is abelian (this is the case when $s \in \mathfrak{g}$ is regular semisimple), then $h(z) - \tilde{h}(z) = \partial S(z)$.

Corollary 2.9. Let \mathfrak{g} be a finite-dimensional Lie algebra with a non-degenerate symmetric invariant bilinear form κ , and let $s \in \mathfrak{g}$ be a semisimple element. Let $\mathcal{V} = S(\mathbb{F}[\partial]\mathfrak{g}) = \mathbb{F}[u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$ (where $\{u_i\}_{i \in I} \subset \mathfrak{g}$ is a basis of \mathfrak{g}), and let us extend κ to a bilinear map $\kappa : \mathfrak{g} \otimes \mathcal{V} \times \mathfrak{g} \otimes \mathcal{V} \rightarrow \mathcal{V}$. Let $U(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]z^{-1}$, $h(z) \in (\mathfrak{h} \otimes \mathcal{V})[[z^{-1}]]$ be a solution of equation (2.2), where $\mathfrak{h} = \text{Ker}(\text{ad } s)$. Given an element $a \in Z(\mathfrak{h}) \setminus Z(\mathfrak{g})$, we have an infinite hierarchy of integrable bi-Hamiltonian equations associated to the bi-Hamiltonian structure (H, K) on \mathcal{V} , defined in (1.19):

$$\frac{du_i}{dt_n} = \sum_{j \in I} H_{ij}(\partial) \frac{\delta f_n}{\delta u_j} = \sum_{j \in I} K_{ij}(\partial) \frac{\delta f_{n+1}}{\delta u_j}, \quad i \in I, n \in \mathbb{Z}_+,$$

where $\int f_n \in \mathcal{V}/\partial\mathcal{V}$ is the coefficient of z^{-n} in $\int f(z) = \int \kappa(a \otimes 1 \mid h(z)) \in (\mathcal{V}/\partial\mathcal{V})[[z^{-1}]]$.

Proof. By Propositions 2.3 and 2.4, the formal power series $F(z) \in (\mathfrak{g} \otimes \mathcal{V})[[z^{-1}]]$, defined in (2.3), satisfies conditions (C1), (C2) and (C3) above. Hence, according to the Lenard-Magri scheme, we only need to check that the local functionals $\int f_n \in \mathcal{V}/\partial\mathcal{V}$, $n \in \mathbb{Z}_+$, are linearly independent. This is obtained by the following simple observations. By the definitions (1.19) of H and K , it is clear that, if $F = a \otimes 1$ with $a \notin Z(\mathfrak{g})$, then $H(F) \neq 0$ and it is of differential order 0; if $F \in \mathfrak{g} \otimes \mathcal{V}$ has differential order $m \in \mathbb{Z}_+$, then necessarily $H(F) \in \mathfrak{g} \otimes \mathcal{V}$ has differential order $m+1$, and if $K(F) \in \mathfrak{g} \otimes \mathcal{V}$ has differential order m , then necessarily $F \in \mathfrak{g} \otimes \mathcal{V}$ has differential order at least m . Hence, by the recursion formula (1.20) we immediately get that the elements $F_n \in \mathfrak{g} \otimes \mathcal{V}$, $n \in \mathbb{F}_+$, have distinct differential orders. In particular, the elements $F_n = \frac{\delta f_n}{\delta u} \in \mathfrak{g} \otimes \mathcal{V}$ are linearly independent, and therefore the elements $\int f_n \in \mathcal{V}/\partial\mathcal{V}$ are linearly independent as well. \square

Example 2.10. The N -wave equation. Let $\mathfrak{g} = \mathfrak{gl}_N$, with the bilinear form $\kappa(A \mid B) = \text{Tr}(AB)$, and let $s = \text{diag}(s_1, \dots, s_N)$ be a diagonal matrix with distinct eigenvalues. Then $\mathfrak{h} = \text{Ker}(\text{ad } s)$ is the abelian subalgebra of diagonal $N \times N$ matrices, and $\mathfrak{h}^\perp = \text{Im}(\text{ad } s)$ consists of $N \times N$ matrices with zeros along the diagonal. We also have $u = \sum_{i,j=1}^N E_{ij} \otimes E_{ji} \in \mathfrak{gl}_N \otimes \mathcal{V}$, where \mathcal{V} is the algebra of differential polynomials generated by \mathfrak{gl}_N . In this case, for $U(z) \in (\mathfrak{h}^\perp \otimes \mathcal{V})[[z^{-1}]]z^{-1}$, there exists a unique $T(z) = \sum_{n \in \mathbb{Z}_+} T_n z^{-n} \in (\mathfrak{gl}_N \otimes \mathcal{V})[[z^{-1}]]$, with $T_0 = 1_N$ and $T_n \in \mathfrak{h}^\perp$ for all $n \geq 1$, such that $e^{\text{ad } U(z)} : \mathfrak{gl}_N \rightarrow \mathfrak{gl}_N$ coincides with conjugation by $T(z)$. Hence, equation (2.2) reduces, in this case, to finding $T(z)$ as above, and $h(z) = \sum_{n \in \mathbb{Z}_+} h_n z^{-n} \in \mathfrak{h} \otimes \mathcal{V}[[z^{-1}]]$, such that

$$T(z)(\partial + u + zs \otimes 1) = (\partial + zs \otimes 1 + h(z))T(z).$$

The above equation gives rise to the following recursive formula for $h_n \in \mathfrak{h} \otimes \mathcal{V}$ and $T_{n+1} \in \mathfrak{h}^\perp \otimes \mathcal{V}$ ($n \in \mathbb{Z}_+$):

$$h_n + [s \otimes 1, T_{n+1}] = T_n u - \partial T_n - \sum_{k=0}^{n-1} h_k T_{n-k}.$$

Clearly, the above equation determines h_n and T_{n+1} uniquely. The first few terms in the recursion are given by (in the sums below the terms with zero denominator are dropped):

$$\begin{aligned} h_0 &= \sum_k E_{kk} \otimes E_{kk}, & T_1 &= \sum_{i,j} E_{ij} \otimes \frac{E_{ji}}{s_i - s_j}, \\ h_1 &= \sum_k E_{kk} \otimes \left(\sum_\alpha \frac{E_{k\alpha} E_{\alpha k}}{s_k - s_\alpha} \right), & T_2 &= \sum_{i,j} E_{ij} \otimes \left(\sum_k \frac{E_{jk} E_{ki}}{(s_i - s_j)(s_i - s_k)} - \frac{E'_{ji} + E_{ji} E_{ii}}{(s_i - s_j)^2} \right), \\ h_2 &= \sum_k E_{kk} \otimes \left(\sum_{\alpha, \beta} \frac{E_{k\alpha} E_{\alpha\beta} E_{\beta k}}{(s_k - s_\alpha)(s_k - s_\beta)} - \sum_\alpha \frac{E_{k\alpha} E'_{\alpha k} + E_{k\alpha} E_{\alpha k} E_{kk}}{(s_k - s_\alpha)^2} \right). \end{aligned}$$

The integrable hierarchy associated to the non-scalar element $a = \text{diag}(a_1, \dots, a_N) \in Z(\mathfrak{h}) = \mathfrak{h}$ is defined in terms of the Hamiltonian functionals in involution $\int f_n = \int \text{Tr}((a \otimes 1)h_n)$, $n \in \mathbb{Z}_+$. The first few elements are (again the terms with zero denominator are dropped from the sums):

$$\begin{aligned} \int f_0 &= \int \sum_k a_k E_{kk}, & \int f_1 &= \int \sum_{k,\alpha} \frac{a_k E_{k\alpha} E_{\alpha k}}{s_k - s_\alpha}, \\ \int f_2 &= \int \sum_{\alpha,\beta,k} \frac{a_k E_{k\alpha} E_{\alpha\beta} E_{\beta k}}{(s_k - s_\alpha)(s_k - s_\beta)} - \sum_{\alpha,k} \frac{a_k (E_{k\alpha} E'_{\alpha k} + E_{k\alpha} E_{\alpha k} E_{kk})}{(s_k - s_\alpha)^2}. \end{aligned}$$

The corresponding hierarchy of Hamiltonian equations is $\frac{du}{dt_n} = H(F_n) = \partial F_n + [u, F_n]$, $n \in \mathbb{Z}_+$, where $F(z) = \sum_{n \in \mathbb{Z}_+} F_n z^{-n} = T(z)^{-1}(a \otimes 1)T(z)$ (see equation (2.3)). In particular, $F_0 = a \otimes 1$ and $F_1 = [a \otimes 1, T_1] = \sum_{i \neq j} \frac{a_i - a_j}{s_i - s_j} E_{ij} \otimes E_{ji} \in \mathfrak{gl}_N \otimes \mathcal{V}$. Hence, the first two equations of the hierarchy are ($1 \leq i, j \leq N$):

$$\frac{dE_{ij}}{dt_0} = (a_i - a_j)E_{ij}, \quad \frac{dE_{ij}}{dt_1} = \gamma_{ij} E'_{ij} + \sum_k (\gamma_{ik} - \gamma_{kj}) E_{ik} E_{kj},$$

where $\gamma_{ij} = \frac{a_i - a_j}{s_i - s_j}$ for $i \neq j$ and $\gamma_{ij} = 0$ for $i = j$. The last equation is known as the N -wave equation.

3. CLASSICAL \mathcal{W} -ALGEBRAS

In this section we give the definition of classical \mathcal{W} -algebras in the language of Poisson vertex algebras. We also show how this definition is related to the original definition of Drinfeld and Sokolov [DS85]. We thus obtain a bi-Hamiltonian structure for classical \mathcal{W} -algebras, that we will use in the next section to apply successfully the Lenard-Magri scheme of integrability.

3.1. Setup. Throughout the rest of the paper we make the following assumptions.

Let \mathfrak{g} be a reductive finite-dimensional Lie algebra over the field \mathbb{F} with a non-degenerate symmetric invariant bilinear form κ , and let $f \in \mathfrak{g}$ be a non-zero nilpotent element. By the Jacobson-Morozov Theorem [CMG93, Theorem 3.3.1], it is possible to embed f in an \mathfrak{sl}_2 -triple $(f, h = 2x, e) \subset \mathfrak{g}$. Then we have the $\text{ad } x$ -eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_i. \quad (3.1)$$

Clearly, $f \in \mathfrak{g}_{-1}$, $h \in \mathfrak{g}_0$ and $e \in \mathfrak{g}_1$.

There is a well-known skew-symmetric bilinear form ω on $\mathfrak{g}_{\frac{1}{2}}$ defined by

$$\omega(a, b) = \kappa(f \mid [a, b]), \quad a, b \in \mathfrak{g}_{\frac{1}{2}},$$

which is non-degenerate since $\text{ad } f : \mathfrak{g}_{\frac{1}{2}} \rightarrow \mathfrak{g}_{-\frac{1}{2}}$ is bijective. Fix an isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ (with respect to ω) and denote by $\mathfrak{l}^{\perp\omega} = \{a \in \mathfrak{g}_{\frac{1}{2}} \mid \omega(a, b) = 0 \text{ for all } b \in \mathfrak{l}\} \subset \mathfrak{g}_{\frac{1}{2}}$ its symplectic complement with respect to ω . We consider the following nilpotent subalgebras of \mathfrak{g} :

$$\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{g}_{\geq 1} \subset \mathfrak{n} = \mathfrak{l}^{\perp\omega} \oplus \mathfrak{g}_{\geq 1}, \quad (3.2)$$

where $\mathfrak{g}_{\geq 1} = \bigoplus_{i \geq 1} \mathfrak{g}_i$.

Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$, where \mathfrak{p} is an arbitrary subspace of \mathfrak{g} complementary to \mathfrak{m} . Since κ is non-degenerate, we also have the corresponding decomposition with the orthogonal complements $\mathfrak{g} = \mathfrak{p}^{\perp} \oplus \mathfrak{m}^{\perp}$. Hence, identifying $\mathfrak{g} \simeq \mathfrak{g}^*$ via κ , we get the isomorphisms $\mathfrak{p}^* \simeq \mathfrak{g}/\mathfrak{p}^{\perp} \simeq \mathfrak{m}^{\perp}$. Let $\{q_i\}_{i \in P}$ be a basis of \mathfrak{p} , and let $\{q^i\}_{i \in P}$ be the dual (with respect to κ) basis of \mathfrak{m}^{\perp} , namely, such that $\kappa(q^j \mid q_i) = \delta_{ij}$. These dual bases are equivalently defined by the completeness relations

$$\sum_{j \in P} \kappa(q^j \mid a) q_j = \pi_{\mathfrak{p}} a, \quad \sum_{j \in P} \kappa(a \mid q_j) q^j = \pi_{\mathfrak{m}^{\perp}} a \quad \text{for all } a \in \mathfrak{g}, \quad (3.3)$$

where $\pi_{\mathfrak{p}} : \mathfrak{g} \rightarrow \mathfrak{p}$ and $\pi_{\mathfrak{m}^{\perp}} : \mathfrak{g} \rightarrow \mathfrak{m}^{\perp}$ are the projection maps with kernels \mathfrak{m} and \mathfrak{p}^{\perp} respectively.

Finally, we fix an element $s \in \text{Ker}(\text{ad } \mathfrak{n}) \subset \mathfrak{g}$. In the next section, when applying the Lenard-Magri scheme of integrability, we will need some further assumptions on the element s (see Section 4.2).

3.2. Definition of classical \mathcal{W} -algebras. Let us consider the affine PVA $\mathcal{V}(\mathfrak{g}) = \mathcal{V}(\mathfrak{g}, \kappa, zs)$, where $z \in \mathbb{F}$, from Example 1.4. As a differential algebra, it is $\mathcal{V}(\mathfrak{g}) = S(\mathbb{F}[\partial]\mathfrak{g})$, and the λ -bracket on it is given by

$$\{a_\lambda b\}_z = [a, b] + \kappa(a \mid b)\lambda + z\kappa(s \mid [a, b]) \quad , \quad a, b \in \mathfrak{g}, \quad (3.4)$$

and extended to $\mathcal{V}(\mathfrak{g})$ by the Master Formula (1.9). Note that, since, by assumption, $[s, \mathfrak{n}] = 0$, $\mathbb{F}[\partial]\mathfrak{n} \subset \mathcal{V}(\mathfrak{g})$ is a Lie conformal subalgebra (see Definition 1.5), with the λ -bracket $\{a_\lambda b\}_z = [a, b]$, $a, b \in \mathfrak{n}$ (it is independent on z).

Consider the differential subalgebra $\mathcal{V}(\mathfrak{p}) = S(\mathbb{F}[\partial]\mathfrak{p})$ of $\mathcal{V}(\mathfrak{g})$, and denote by $\rho : \mathcal{V}(\mathfrak{g}) \rightarrow \mathcal{V}(\mathfrak{p})$, the differential algebra homomorphism defined on generators by

$$\rho(a) = \pi_{\mathfrak{p}}(a) + \kappa(f \mid a), \quad a \in \mathfrak{g}. \quad (3.5)$$

Lemma 3.1. (a) For every $a \in \mathfrak{n}$ and $g \in \mathcal{V}(\mathfrak{m}) = S(\mathbb{F}[\partial]\mathfrak{m}) \subset \mathcal{V}(\mathfrak{g})$, we have $\rho\{a_\lambda g\}_z = 0$.

(b) For every $a \in \mathfrak{n}$ and $g \in \mathcal{V}(\mathfrak{g})$, we have $\rho\{a_\lambda \rho(g)\}_z = \rho\{a_\lambda g\}_z$.

(c) We have a representation of the Lie conformal algebra $\mathbb{F}[\partial]\mathfrak{n}$ on the differential subalgebra $\mathcal{V}(\mathfrak{p}) \subset \mathcal{V}(\mathfrak{g})$ given by ($a \in \mathfrak{n}$, $g \in \mathcal{V}(\mathfrak{p})$):

$$a_\lambda^\rho g = \rho\{a_\lambda g\}_z \quad (3.6)$$

(note that the RHS is independent of z since, by assumption, $s \in \text{Ker}(\text{ad } \mathfrak{n})$).

(d) The λ -action of $\mathbb{F}[\partial]\mathfrak{n}$ on $\mathcal{V}(\mathfrak{p})$ given by (3.6) is by conformal derivations (see Definition 1.5(c)).

Proof. To prove (a), we can use the Master Formula (1.9) to reduce to the case when $g \in \mathfrak{m}$. In this case, (a) is immediate since, by the definitions (3.2) of \mathfrak{m} and \mathfrak{n} , we have $[\mathfrak{m}, \mathfrak{n}] \subset \mathfrak{m}$ and $\kappa(f \mid [\mathfrak{m}, \mathfrak{n}]) = 0$. Next, let us prove part (b). Since, by construction, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$, we have $\mathcal{V}(\mathfrak{g}) = \mathcal{V}(\mathfrak{m}) \otimes \mathcal{V}(\mathfrak{p})$. Part (b) then follows immediately by part (a) and the left Leibniz rule (1.5). As for parts (c) and (d), clearly the λ -action (3.6) satisfies sesquilinearity and the Leibniz rule, since ρ is a differential algebra homomorphism. We are left to prove the Jacobi identity for this λ -action. For $a, b \in \mathfrak{n}$ and $g \in \mathcal{V}(\mathfrak{p})$ we have, by part (b),

$$a_\lambda^\rho (b_\mu^\rho g) - b_\mu^\rho (a_\lambda^\rho g) = \rho\{a_\lambda \{b_\mu g\}_z\}_z - \rho\{b_\mu \{a_\lambda g\}_z\}_z = \rho\{\{a_\lambda b\}_z \lambda_{\lambda+\mu} g\}_z = \{a_\lambda b\}_z \lambda_{\lambda+\mu}^\rho g.$$

□

We let $\mathcal{W} \subset \mathcal{V}(\mathfrak{p})$ be the subspace killed by the Lie conformal algebra action of $\mathbb{F}[\partial]\mathfrak{n}$:

$$\mathcal{W} = \mathcal{V}(\mathfrak{p})^{\mathbb{F}[\partial]\mathfrak{n}} = \{g \in \mathcal{V}(\mathfrak{p}) \mid a_\lambda^\rho g = 0 \text{ for all } a \in \mathfrak{n}\}. \quad (3.7)$$

Lemma 3.2. (a) $\mathcal{W} \subset \mathcal{V}(\mathfrak{p})$ is a differential subalgebra.

(b) For every $g \in \mathcal{W}$ and $h \in \mathcal{V}(\mathfrak{m})$, we have $\rho\{g_\lambda h\}_z = \rho\{h_\lambda g\}_z = 0$.

(c) For every $g \in \mathcal{W}$ and $h \in \mathcal{V}(\mathfrak{g})$, we have $\rho\{g_\lambda \rho(h)\}_z = \rho\{g_\lambda h\}_z$, and $\rho\{\rho(h)_\lambda g\}_z = \rho\{h_\lambda g\}_z$.

(d) For every $g, h \in \mathcal{W}$, we have $\rho\{g_\lambda h\}_z \in \mathbb{F}[\lambda] \otimes \mathcal{W}$.

(e) The map $\{\cdot_\lambda \cdot\}_{z, \rho} : \mathcal{W} \otimes \mathcal{W} \rightarrow \mathbb{F}[\lambda] \otimes \mathcal{W}$ given by

$$\{g_\lambda h\}_{z, \rho} = \rho\{g_\lambda h\}_z \quad (3.8)$$

defines a PVA structure on \mathcal{W} .

Proof. Part (a) follows from the fact that the λ -action (3.6) of $\mathbb{F}[\partial]\mathfrak{n}$ on $\mathcal{V}(\mathfrak{p})$ is by conformal derivations. As for part (b), we can use the Master Formula (1.9) to reduce to the case when $h \in \mathfrak{m}$, and in this case the statement is obvious by the definition of \mathcal{W} (and the fact that $\mathfrak{m} \subset \mathfrak{n}$). Since $\mathcal{V}(\mathfrak{g}) = \mathcal{V}(\mathfrak{m}) \otimes \mathcal{V}(\mathfrak{p})$, part (c) follows immediately by part (b) and the left and right Leibniz rules (1.5)-(1.6). Next, let us prove part (d). For $a \in \mathfrak{n}$ and $g, h \in \mathcal{W}$,

$$\begin{aligned} \rho\{a_\lambda \rho\{g_\mu h\}_z\}_z &= \rho\{a_\lambda \{g_\mu h\}_z\}_z = \rho\{\{a_\lambda g\}_z \lambda_{\lambda+\mu} h\}_z + \rho\{g_\mu \{a_\lambda h\}_z\}_z \\ &= \rho\{\rho\{a_\lambda g\}_z \lambda_{\lambda+\mu} h\}_z + \rho\{g_\mu \rho\{a_\lambda h\}_z\}_z = 0. \end{aligned}$$

In the first equality we used Lemma 3.1(b), while in the third equality we used part (c). It follows that $\rho\{g_\mu h\}_z$ lies in $\mathbb{F}[\mu] \otimes \mathcal{W}$, proving (d). Finally, let us prove part (e). Since ρ is a differential algebra homomorphism, the λ -bracket (3.8) obviously satisfies sesquilinearity, skewsymmetry, and the left and right Leibniz rules. We are left to check the Jacobi identity. For $g, h, k \in \mathcal{W}$ we have, by part (c),

$$\begin{aligned} \{h_\lambda \{k_\mu g\}_{z, \rho}\}_{z, \rho} &= \rho\{h_\lambda \{k_\mu g\}_z\}_z = \rho\{\{h_\lambda k\}_z \lambda_{\lambda+\mu} g\}_z + \rho\{k_\mu \{h_\lambda g\}_z\}_z \\ &= \{\{h_\lambda k\}_{z, \rho} \lambda_{\lambda+\mu} g\}_{z, \rho} + \{k_\mu \{h_\lambda g\}_{z, \rho}\}_{z, \rho}. \end{aligned}$$

□

Definition 3.3. The classical \mathcal{W} -algebra is the differential algebra \mathcal{W} defined by (3.7) with the PVA structure given by (3.8).

Remark 3.4. The Poisson vertex algebra \mathcal{W} can be constructed in the same way for an arbitrary choice of s in \mathfrak{g} (taking the Lie conformal algebra $\mathbb{F}[\partial]\mathfrak{n} \oplus \mathbb{F}$ of \mathcal{V}). However the differential algebra \mathcal{W} is independent of the choice of $z \in \mathbb{F}$ if and only if $[s, \mathfrak{n}] = 0$. This independence of z will be very important in the next section, where we construct integrable hierarchies of Hamiltonian equations, since there we need to view z as a formal parameter.

Remark 3.5. In literature, the name classical \mathcal{W} -algebra is referred to the Poisson structure corresponding to the case $z = 0$. As we will see the whole family of PVAs \mathcal{W} , parametrized by $z \in \mathbb{F}$, plays an important role in obtaining an integrable hierarchy of Hamiltonian equations associated to the classical \mathcal{W} -algebra.

Recall that we fixed a basis $\{q_i\}_{i \in P}$ of \mathfrak{p} and the dual basis $\{q^i\}_{i \in P}$ of \mathfrak{m}^\perp . We can find an explicit formula for the λ -bracket in \mathcal{W} as follows. Recalling the Master Formula (1.9) and using (3.5) and the definition (3.4) of the λ -bracket in \mathcal{V} , we get $(g, h \in \mathcal{W})$:

$$\{g_\lambda h\}_{z, \rho} = \{g_\lambda h\}_{H, \rho} - z \{g_\lambda h\}_{K, \rho}, \quad (3.9)$$

where

$$\{g_\lambda h\}_{X, \rho} = \sum_{\substack{i, j \in P \\ m, n \in \mathbb{Z}_+}} \frac{\partial h}{\partial q_j^{(n)}} (\lambda + \partial)^n X_{ji} (\lambda + \partial) (-\lambda - \partial)^m \frac{\partial g}{\partial q_i^{(m)}}, \quad (3.10)$$

for X one of the two matrices $H, K \in \text{Mat}_{k \times k} \mathcal{V}(\mathfrak{p})[\lambda]$ ($k = \#(P)$), given by

$$H_{ji}(\lambda) = \pi_{\mathfrak{p}}[q_i, q_j] + \kappa(q_i | q_j) \lambda + \kappa(f | [q_i, q_j]), \quad K_{ji}(\partial) = \kappa(s | [q_j, q_i]), \quad (3.11)$$

for $i, j \in P$.

Recall that a $k \times k$ matrix with entries in $\mathcal{V}(\mathfrak{p})[\lambda]$ corresponds, via (1.12), to a linear map $\mathfrak{p} \otimes \mathcal{V}(\mathfrak{p}) \rightarrow \mathfrak{p}^* \otimes \mathcal{V}(\mathfrak{p})$, and that we can identify $\mathfrak{p}^* \simeq \mathfrak{m}^\perp$ via the bilinear form κ . Therefore, we can describe the above matrices H and K as the following linear maps $\mathfrak{p} \otimes \mathcal{V}(\mathfrak{p}) \rightarrow \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$:

$$H(a \otimes g) = \sum_{i \in P} \pi_{\mathfrak{m}^\perp}[q^i, a] \otimes q_i g + \pi_{\mathfrak{m}^\perp}[f, a] \otimes g + \pi_{\mathfrak{m}^\perp}(a) \otimes \partial g, \quad K(a \otimes g) = \pi_{\mathfrak{m}^\perp}[a, s] \otimes g, \quad (3.12)$$

for every $a \in \mathfrak{p}$ and $g \in \mathcal{V}(\mathfrak{p})$.

Note that, even though the λ -bracket (3.9) on the PVA \mathcal{W} is formally associated to the matrices H and K in (3.11) via the Master Formula (1.9), H and K are NOT Hamiltonian structures (on $\mathcal{V}(\mathfrak{p})$). Indeed, $\mathcal{V}(\mathfrak{p})$ is not a PVA, namely the λ -bracket $\{\cdot_\lambda \cdot\}_{z, \rho}$ on $\mathcal{V}(\mathfrak{p})$ (given by the same formula (3.9)) is not a PVA λ -bracket.

Remark 3.6. The classical \mathcal{W} -algebra can be equivalently defined, without fixing a complementary subspace $\mathfrak{p} \subset \mathfrak{g}$ of \mathfrak{m} , via the so called “classical Hamiltonian reduction” (see [DSK06]). The general construction is as follows. Let \mathcal{V} be a Poisson vertex algebra, let $I \subset \mathcal{V}$ be a differential ideal of \mathcal{V} (viewed as differential algebra), and let R be a Lie conformal algebra acting on \mathcal{V} by conformal derivations of the product and λ -bracket (for example, R is a Lie conformal subalgebra of \mathcal{V}), such that $\{R_\lambda I\} \subset \mathbb{F}[\lambda] \otimes I$. The corresponding *classical Hamiltonian reduction* is defined as the differential algebra

$$\mathcal{W}(\mathcal{V}, R, I) = (\mathcal{V}/I)^R = \{f + I \mid \{a_\lambda f\} \in \mathbb{F}[\lambda] \otimes I \text{ for all } a \in R\},$$

endowed with the λ -bracket $\{f + I_\lambda g + I\} = \{f_\lambda g\} + \mathbb{F}[\lambda] \otimes I$. It is not hard to show that this λ -bracket is well defined. The classical \mathcal{W} -algebra is obtained by taking $\mathcal{V} = \mathcal{V}(\mathfrak{g}, \kappa, s)$, $R = \mathbb{F}[\partial]\mathfrak{n}$, and

$$I = \text{Ker}(\rho) = (m - \kappa(f | m) \mid m \in \mathfrak{m}) \subset \mathcal{V}(\mathfrak{g}),$$

the differential ideal generated by the elements $m - \kappa(f | m)$, for $m \in \mathfrak{m}$ (note that $\text{Ker} \rho$ is independent of the choice of \mathfrak{p}). To see this, let also

$$\widetilde{\mathcal{W}} = \{g \in \mathcal{V}(\mathfrak{g}) \mid \{a_\lambda g\}_z \in \mathbb{F}[\lambda] \otimes \text{Ker} \rho \text{ for all } a \in \mathfrak{n}\} \subset \mathcal{V}(\mathfrak{g}). \quad (3.13)$$

Since $[s, \mathfrak{n}] = 0$, the space $\widetilde{\mathcal{W}}$ is independent of z . Clearly, the map $\rho : \mathcal{V}(\mathfrak{g}) \rightarrow \mathcal{V}(\mathfrak{p})$ induces differential algebra isomorphism $\mathcal{V}(\mathfrak{g})/\text{Ker} \rho \simeq \mathcal{V}(\mathfrak{p})$, which restricts to a differential algebra isomorphism $\widetilde{\mathcal{W}}/\text{Ker} \rho \simeq \mathcal{W}$.

Remark 3.7. The PVA \mathcal{W} was constructed in [DSK06] as a quasiclassical limit of a family of vertex algebras, obtained by a cohomological construction in [KW04]. The isomorphism of this construction with the construction in the present paper via classical Hamiltonian reduction is proved in [Suh12].

3.3. Gauge transformations and Drinfeld-Sokolov approach to classical \mathcal{W} -algebras. In this section we show that the definition of the classical \mathcal{W} -algebra given in Section 3.2 is equivalent to the original definition of Drinfeld and Sokolov [DS85], given in terms of gauge invariance.

Recall from Section 2 the definition of the Lie algebra $\mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V}(\mathfrak{g}))$. The subspace $\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$ is clearly a Lie subalgebra. Let

$$q = \sum_{i \in P} q^i \otimes q_i \in \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p}). \quad (3.14)$$

Note that $q = (\pi_{\mathfrak{m}^\perp} \otimes 1)u$, where $u \in \mathfrak{g} \otimes \mathcal{V}(\mathfrak{g})$ was defined in Proposition 2.1. Let

$$L = \partial + q + f \otimes 1 \in \mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})).$$

A *gauge transformation* is, by definition, a change of variables formula $q \mapsto q^A \in \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$, for $A \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$, given by

$$e^{\text{ad } A} L = \partial + q^A + f \otimes 1. \quad (3.15)$$

In [DS85], Drinfeld and Sokolov defined the classical \mathcal{W} -algebra as the subspace $\mathcal{W} \subset \mathcal{V}(\mathfrak{p})$ consisting of *gauge invariant* differential polynomials g , that is, such that $g(q^A) = g(q)$ for every $A \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$. Here and further we use the following notation: for $g \in \mathcal{V}(\mathfrak{p})$ and $r = \sum_i q^i \otimes r_i \in \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$, we let $g(r)$ be the differential polynomial in q_1, \dots, q_k obtained replacing $q_i^{(m)}$ by $\partial^m r_i$ in the differential polynomial g .

In this section we will prove that the space of gauge invariant polynomials coincides with the space \mathcal{W} defined in (3.7). The key observation is that the action of the gauge group $g \mapsto g(q^A) \in \mathcal{V}(\mathfrak{p})$ is obtained by exponentiating the Lie conformal algebra action of $\mathbb{F}[\partial]\mathfrak{n}$ on $\mathcal{V}(\mathfrak{p})$ given by (3.6). This is stated in the following

Theorem 3.8. *For every $a \otimes h \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ and $g \in \mathcal{V}(\mathfrak{p})$, we have*

$$g(q^{a \otimes h}) = \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{n!} (a_{\lambda_1}^\rho \dots a_{\lambda_n}^\rho g) (|_{\lambda_1=\partial} h) \dots (|_{\lambda_n=\partial} h), \quad (3.16)$$

where, for a polynomial $p(\lambda_1, \dots, \lambda_n) = \sum c \lambda_1^{i_1} \dots \lambda_n^{i_n}$, we denote $p(\lambda_1, \dots, \lambda_n) (|_{\lambda_1=\partial} h_1) \dots (|_{\lambda_n=\partial} h_n) = \sum c (\partial^{i_1} h_1) \dots (\partial^{i_n} h_n)$.

Proof. First, by Lemma 3.1(b), the RHS of equation (3.16) is

$$\sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{n!} \rho \{ a_{\lambda_1} \dots \{ a_{\lambda_n} g \}_z \dots \}_z (|_{\lambda_1=\partial} h) \dots (|_{\lambda_n=\partial} h).$$

Next, we expand the LHS of equation (3.16) in Taylor series, using the definition (3.15) of $q^{a \otimes h}$:

$$\begin{aligned} g(q^{a \otimes h}) &= g \left(q + \sum_{n \geq 1} \frac{1}{n!} (\text{ad } a \otimes h)^n (\partial + f \otimes 1 + q) \right) = \sum_{\substack{s \in \mathbb{Z}_+, i_1, \dots, i_s \in P \\ m_1, \dots, m_s \in \mathbb{Z}_+ \\ n_1, \dots, n_s \geq 1}} \frac{1}{s! n_1! \dots n_s!} \\ &\times \frac{\partial^s g}{\partial q_{i_1}^{(m_1)} \dots \partial q_{i_s}^{(m_s)}} (\partial^{m_1} (\text{ad } a \otimes h)^{n_1} (\partial + f \otimes 1 + q))_{i_1} \dots (\partial^{m_s} (\text{ad } a \otimes h)^{n_s} (\partial + f \otimes 1 + q))_{i_s}. \end{aligned}$$

Combining the above equations, we deduce that equation (3.16) is equivalent to (for every $N \in \mathbb{Z}_+$):

$$\begin{aligned} \rho \{ a_{\lambda_1} \dots \{ a_{\lambda_N} g \}_z \dots \}_z (|_{\lambda_1=\partial} h) \dots (|_{\lambda_N=\partial} h) &= \sum_{\substack{s \in \mathbb{Z}_+, i_1, \dots, i_s \in P \\ m_1, \dots, m_s \in \mathbb{Z}_+ \\ n_1, \dots, n_s \geq 1 \\ (n_1 + \dots + n_s = N)}} \frac{N!}{s! n_1! \dots n_s!} (-1)^N \\ &\times \frac{\partial^s g}{\partial q_{i_1}^{(m_1)} \dots \partial q_{i_s}^{(m_s)}} (\partial^{m_1} (\text{ad } a \otimes h)^{n_1} (\partial + f \otimes 1 + q))_{i_1} \dots (\partial^{m_s} (\text{ad } a \otimes h)^{n_s} (\partial + f \otimes 1 + q))_{i_s}. \end{aligned} \quad (3.17)$$

We start by proving equation (3.17) when $g = q_i$, for every $i \in P$. Namely, we need to prove that, for every $n \geq 1$, we have

$$\rho \{ a_{\lambda_1} \dots \{ a_{\lambda_n} q_i \}_z \dots \}_z (|_{\lambda_1=\partial} h) \dots (|_{\lambda_n=\partial} h) = (-1)^n ((\text{ad } a \otimes h)^n (\partial + f \otimes 1 + q))_i. \quad (3.18)$$

By the second completeness relation (3.3) and the invariance of the bilinear form κ , we have

$$\begin{aligned} (-1)^n ((\text{ad } a \otimes h)^n (\partial + f \otimes 1 + q))_i &= (-1)^n \kappa((\text{ad } a \otimes h)^n (\partial + f \otimes 1 + q) | q_i \otimes 1) \\ &= \delta_{n,1} \kappa(a | q_i) \partial h + \kappa(f | (\text{ad } a)^n(q_i)) h^n + \pi_{\mathfrak{p}}(\text{ad } a)^n(q_i) h^n, \end{aligned}$$

which is the same as the LHS of (3.18). Note that, in view of the above computation, the LHS of (3.18) is the same as $((a_\partial^\rho)^n q_i) \rightarrow h^n$, where, as usual, the arrow means that ∂ should be moved to the right (to act on h^n).

In view of (3.18) (and the above observation), equation (3.17) can be rewritten as follows

$$\begin{aligned} & \rho\{a_{\lambda_1} \dots \{a_{\lambda_N} g\}_z \dots\}_z (|_{\lambda_1=\partial} h) \dots (|_{\lambda_N=\partial} h) \\ &= \sum_{\substack{s \in \mathbb{Z}_+, i_1, \dots, i_s \in P \\ m_1, \dots, m_s \in \mathbb{Z}_+ \\ n_1, \dots, n_s \geq 1 \\ (n_1 + \dots + n_s = N)}} \frac{N!}{s!n_1! \dots n_s!} \frac{\partial^s g}{\partial q_{i_1}^{(m_1)} \dots \partial q_{i_s}^{(m_s)}} \left(\partial^{m_1} ((a_\partial^\rho)^{n_1} q_{i_1}) \rightarrow h^{n_1} \right) \dots \left(\partial^{m_s} ((a_\partial^\rho)^{n_s} q_{i_s}) \rightarrow h^{n_s} \right). \end{aligned} \quad (3.19)$$

Let us denote the two sides of equation (3.19) by $LHS_N(g)$ and $RHS_N(g)$. The identity $LHS_N(q_i) = RHS_N(q_i)$ for every $i \in P$ and every $N \in \mathbb{Z}_+$ is given by the above observations. Moreover, it is easy to check that $LHS_N(\partial g) = \partial LHS_N(g)$ and, using equation (1.2) s times, we also have that $RHS_N(\partial g) = \partial RHS_N(g)$. Furthermore, it is not difficult to prove, using the left Leibniz rule (1.5), that $LHS_N(g)$ satisfies the functional equation

$$LHS_N(g_1 g_2) = \sum_{n=0}^N \binom{N}{n} LHS_n(g_1) LHS_{N-n}(g_2).$$

In order to prove that (3.19) holds for every $g \in \mathcal{V}(\mathfrak{p})$, it suffices to show that $RHS_N(g)$ satisfies the same functional equation. We have, by the Leibniz rule for partial derivatives,

$$\begin{aligned} RHS_N(g_1 g_2) &= \sum_{\substack{s \in \mathbb{Z}_+, i_1, \dots, i_s \in P \\ m_1, \dots, m_s \in \mathbb{Z}_+ \\ n_1, \dots, n_s \geq 1 \\ (n_1 + \dots + n_s = N)}} \frac{N!}{s!n_1! \dots n_s!} \frac{\partial^s g_1 g_2}{\partial q_{i_1}^{(m_1)} \dots \partial q_{i_s}^{(m_s)}} \left(\partial^{m_1} ((a_\partial^\rho)^{n_1} q_{i_1}) \rightarrow h^{n_1} \right) \dots \left(\partial^{m_s} ((a_\partial^\rho)^{n_s} q_{i_s}) \rightarrow h^{n_s} \right) \\ &= \sum_{\substack{s \in \mathbb{Z}_+, i_1, \dots, i_s \in P \\ m_1, \dots, m_s \in \mathbb{Z}_+ \\ n_1, \dots, n_s \geq 1 \\ (n_1 + \dots + n_s = N)}} \frac{N!}{s!n_1! \dots n_s!} \sum_{a=0}^s \binom{s}{a} \frac{\partial^a g_1}{\partial q_{i_1}^{(m_1)} \dots \partial q_{i_a}^{(m_a)}} \frac{\partial^{s-a} g_2}{\partial q_{i_{a+1}}^{(m_{a+1})} \dots \partial q_{i_s}^{(m_s)}} \left(\partial^{m_1} ((a_\partial^\rho)^{n_1} q_{i_1}) \rightarrow h^{n_1} \right) \dots \\ &\quad \dots \left(\partial^{m_s} ((a_\partial^\rho)^{n_s} q_{i_s}) \rightarrow h^{n_s} \right) = \sum_{n=0}^N \binom{N}{n} RHS_n(g_1) RHS_{N-n}(g_2). \end{aligned}$$

□

Remark 3.9. The gauge transformation $g(q) \mapsto g(q^A)$ is not a group action on $\mathcal{V}(\mathfrak{p})$. In view of Theorem 3.8, it is rather a “Lie conformal group” action.

As immediate consequence of Theorem 3.8 we have the following result.

Corollary 3.10. *The space of gauge invariant differential polynomials $g \in \mathcal{V}(\mathfrak{p})$ coincides with the differential algebra \mathcal{W} defined in (3.7).*

3.4. Generators of the classical \mathcal{W} -algebra. Using the description of the classical \mathcal{W} -algebras in terms of gauge invariance, we will prove, following the ideas of Drinfeld and Sokolov, that the differential algebra \mathcal{W} is an algebra of differential polynomials in $r = \dim \text{Ker}(\text{ad } f)$ variables, and we will provide an algorithm to find explicit generators. A cohomological proof of this for quantum \mathcal{W} -algebras, which also works for classical \mathcal{W} -algebras, was given in [KW04, DSK06].

Note that, by the definitions (3.2) of \mathfrak{m} and \mathfrak{n} , we have $[f, \mathfrak{n}] \subset \mathfrak{m}^\perp$. Furthermore, from representation theory of \mathfrak{sl}_2 , we know that $\text{ad } f : \mathfrak{n} \rightarrow \mathfrak{m}^\perp$ is an injective map. Fix a subspace $V \subset \mathfrak{m}^\perp$ complementary to $[f, \mathfrak{n}]$, compatible with the direct sum decomposition (3.1): $V = \bigoplus_{i \geq 0} V_i$, where $V_i \subset \mathfrak{m}^\perp \cap \mathfrak{g}_i$ is a subspace complementary to $[f, \mathfrak{n} \cap \mathfrak{g}_{i+1}]$. Clearly, $\dim(V) = \dim \text{Ker}(\text{ad } f)$. (By representation theory of \mathfrak{sl}_2 we can choose, for example, $V = \text{Ker}(\text{ad } e)$).

Before stating the main result of this section, we introduce some important gradings. In the algebra of differential polynomials $\mathcal{V}(\mathfrak{p})$ we have the usual *polynomial grading*, $\mathcal{V}(\mathfrak{p}) = \bigoplus_{n \in \mathbb{Z}_+} \mathcal{V}(\mathfrak{p})(n)$. For a homogeneous polynomial $g \in \mathcal{V}(\mathfrak{p})$, we denote by $\deg(g)$ its degree, and for an arbitrary element we let $g(n)$ be its homogeneous component of degree n . Also, we extend this decomposition to $\mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$

by looking only at the second factor in the tensor products. We thus have the polynomial degree decomposition

$$\mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p}) = \bigoplus_{n \in \mathbb{Z}_+} \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})(n). \quad (3.20)$$

The algebra \mathfrak{g} , as well as its subspace \mathfrak{m}^\perp and its subalgebra $\mathfrak{n} \subset \mathfrak{m}^\perp$, has the decomposition (3.1) by $\text{ad } x$ -eigenspaces. We extend this grading to the Lie algebra $\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$ by looking only at the first factor in the tensor product:

$$\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}) = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_i \otimes \mathcal{V}(\mathfrak{p}). \quad (3.21)$$

For a homogeneous element $X \in \mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$ we denote by $\delta_x(X)$ its $\text{ad } x$ -eigenvalue, and for an arbitrary element we let $X[i]$ be its homogeneous component of $\text{ad } x$ -eigenvalue i . In the algebra of differential polynomials $\mathcal{V}(\mathfrak{p})$ we introduce a second grading, which we call *conformal weight*, and we denote by Δ , defined as follows. For a monomial $g = a_1^{(m_1)} \dots a_s^{(m_s)}$, product of derivatives of elements $a_i \in \mathfrak{p}$ homogeneous with respect to the $\text{ad } x$ -eigenspace decomposition (3.1), we define its conformal weight as

$$\Delta(g) = s - \delta_x(a_1) - \dots - \delta_x(a_s) + m_1 + \dots + m_s. \quad (3.22)$$

As we will see below, this grading restricts to the conformal weight w.r.t. an explicitly defined Virasoro field of \mathcal{W} , hence the name “conformal weight”. Thus we get the conformal weight space decomposition $\mathcal{V}(\mathfrak{p}) = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathcal{V}(\mathfrak{p})\{i\}$. Finally, we define a grading of the Lie algebra $\mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}))$, which we call *weight* and denote by wt , as follows. We let $\text{wt}(\partial) = 1$, and, for $a \otimes g \in \mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$, we let $\text{wt}(a \otimes g) = -\delta_x(a) + \Delta(g)$. We thus get the corresponding weight space decomposition

$$\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}) = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} (\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}))\{i\}, \quad (3.23)$$

where $(\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}))\{i\} = \bigoplus_j \mathfrak{g}_j \otimes (\mathcal{V}(\mathfrak{p})\{i+j\})$. It is immediate to check that this is indeed a Lie algebra grading of the Lie algebra $\mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}))$.

Lemma 3.11. *Assume that $\mathfrak{g}^f \subset \mathfrak{p}$ (this is the case, for example, if $\mathfrak{p} \subset \mathfrak{g}$ is compatible with the $\text{ad } x$ -eigenspace decomposition (3.1)). Consider the direct sum decomposition $\mathfrak{m}^\perp = [f, \mathfrak{n}] \oplus V$, and let $\{v^j\}_{j \in P}$ be a basis of \mathfrak{m}^\perp , such that $\{v^j\}_{j \in J}$ is a basis of V and $\{v^j\}_{j \in P \setminus J}$ is a basis of $[f, \mathfrak{n}]$. Consider the dual (with respect to κ) basis $\{v_j\}_{j \in P}$ of \mathfrak{p} . Then $\{v_j\}_{j \in J}$ is a basis of \mathfrak{g}^f .*

Proof. First note that, if $\mathfrak{p} \subset \mathfrak{g}$ is compatible with the $\text{ad } x$ -eigenspace decomposition (3.1), then $\mathfrak{p} = \mathfrak{p}_{\frac{1}{2}} \oplus \mathfrak{g}_{\leq 0}$, where $\mathfrak{p}_{\frac{1}{2}} \subset \mathfrak{g}_{\frac{1}{2}}$. Hence, $\mathfrak{g}^f \subset \mathfrak{g}_{\leq 0} \subset \mathfrak{p}$ (proving the statement in parenthesis).

We identify \mathfrak{g} with \mathfrak{g}^* via the bilinear form κ . The decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$ corresponds, via this identification, to the “dual” decomposition $\mathfrak{g} = \mathfrak{p}^\perp \oplus \mathfrak{m}^\perp$, with $\mathfrak{p}^\perp \simeq \mathfrak{m}^*$ and $\mathfrak{m}^\perp \simeq \mathfrak{p}^*$. Similarly, the decomposition $\mathfrak{m}^\perp = [f, \mathfrak{n}] \oplus V$ corresponds, via the same identification, to the “dual” decomposition $\mathfrak{p} = (\mathfrak{p}^\perp \oplus V)^\perp \oplus (\mathfrak{p}^\perp \oplus [f, \mathfrak{n}])^\perp$, with $(\mathfrak{p}^\perp \oplus V)^\perp \simeq [f, \mathfrak{n}]^*$ and $(\mathfrak{p}^\perp \oplus [f, \mathfrak{n}])^\perp \simeq V^*$. Hence, in order to prove the lemma, we only have to show that

$$(\mathfrak{p}^\perp \oplus [f, \mathfrak{n}])^\perp = \mathfrak{g}^f.$$

Since, by assumption, $\mathfrak{g}^f \subset \mathfrak{p}$, the inclusion $\mathfrak{g}^f \subset (\mathfrak{p}^\perp \oplus [f, \mathfrak{n}])^\perp$ is obvious. On the other hand, $(\mathfrak{p}^\perp \oplus [f, \mathfrak{n}])^\perp \simeq V^*$, so that $\dim(\mathfrak{p}^\perp \oplus [f, \mathfrak{n}])^\perp = \dim(V) = \dim(\mathfrak{g}^f)$, proving the claim. \square

Theorem 3.12. (a) *There exists a unique $X \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ such that $w = q^X$ lies in $V \otimes \mathcal{V}(\mathfrak{p})$. Moreover, the element X and w are homogeneous elements of $\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$ with respect to the weight space decomposition (3.23), with weights $\text{wt}(X) = 0$, $\text{wt}(w) = 1$.*

(b) *Let $X \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ be as in (a). Assume that $\mathfrak{p} \subset \mathfrak{g}$ is compatible with the $\text{ad } x$ -eigenspace decomposition (3.1). Let $\{v^j\}_{j \in P}$ be a basis of \mathfrak{m}^\perp consisting of $\text{ad } x$ -eigenvectors, such that $\{v^j\}_{j \in J}$ is a basis of V , and $\{v^j\}_{j \in P \setminus J}$ is a basis of $[f, \mathfrak{n}]$. Let $\{v_j\}_{j \in P}$ be the corresponding dual basis of \mathfrak{p} , so that, by Lemma 3.11, $\{v_j\}_{j \in J}$ is a basis of \mathfrak{g}^f . Then, if we write*

$$q^X = w = \sum_{j \in J} v^j \otimes w_j \in V \otimes \mathcal{V}(\mathfrak{p}),$$

we have that $w_j \in \mathcal{V}(\mathfrak{p})$ is homogeneous with respect to the conformal weight decomposition, of conformal weight $\Delta(w_j) = 1 + \delta_x(v^j)$, and it has the form

$$w_j = v_j + g_j, \quad (3.24)$$

where $g_j = \sum b_1^{(m_1)} \dots b_s^{(m_s)} \in \mathcal{V}(\mathfrak{p})\{1 + \delta_x(v^j)\}$ is a sum with $s + m_1 + \dots + m_s > 1$.

(c) The differential algebra \mathcal{W} is the algebra of differential polynomials in the variables w_1, \dots, w_r .

Proof. Consider the expansion the elements q , X , and w according to $\text{ad } x$ -eigenspace decomposition (3.21) of $\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$: $q = \sum_{i \geq -\frac{1}{2}} q[i]$, where $q[i] \in (\mathfrak{m}^\perp \cap \mathfrak{g}_i) \otimes \mathcal{V}(\mathfrak{p})$, $X = \sum_{i \geq \frac{1}{2}} X[i]$, where $X[i] \in (\mathfrak{n} \cap \mathfrak{g}_i) \otimes \mathcal{V}(\mathfrak{p})$, and $w = \sum_{i \geq 0} w[i]$, where $w[i] \in V_i \otimes \mathcal{V}(\mathfrak{p})$. We want to prove, by induction on $i \geq -\frac{1}{2}$, that the elements $X[i+1] \in (\mathfrak{n} \cap \mathfrak{g}_{i+1}) \otimes \mathcal{V}(\mathfrak{p})$, $i \geq -\frac{1}{2}$, and $w[i] \in V_i \otimes \mathcal{V}(\mathfrak{p})$, $i \geq 0$, are uniquely determined by the equation $q^X = w$, and they are homogeneous of weights $\text{wt}(X[i+1]) = 0$ and $\text{wt}(w[i]) = 1$.

Equating the terms of $\text{ad } x$ -eigenvalue $-\frac{1}{2}$ in both sides of the equation $q^X = w$, we get the equation

$$[f \otimes 1, X[\frac{1}{2}]] = q[-\frac{1}{2}].$$

Since $\text{ad } f$ restricts to a bijection $\mathfrak{g}_{\frac{1}{2}} \xrightarrow{\sim} \mathfrak{g}_{-\frac{1}{2}}$, this uniquely defines $X[\frac{1}{2}] \in (\mathfrak{n} \cap \mathfrak{g}_{\frac{1}{2}}) \otimes \mathcal{V}(\mathfrak{p})$. Moreover, since $q[-\frac{1}{2}]$ is homogeneous of weight 1 and $\text{ad}(f \otimes 1)(\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}))\{i\} \subset (\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}))\{i+1\}$, we get that $\text{wt}(X[\frac{1}{2}]) = 0$.

Next, fix $i \geq 0$ and suppose by induction that we (uniquely) determined all elements $X[j+1] \in (\mathfrak{n} \cap \mathfrak{g}_{j+1}) \otimes \mathcal{V}(\mathfrak{p})$ and $w[j] \in V_j \otimes \mathcal{V}(\mathfrak{p})$ for $j < i$, and that $\text{wt}(X[j+1]) = 0$ and $\text{wt}(w[j]) = 1$. Equating the terms of $\text{ad } x$ -eigenvalue i in both sides of the equation $q^X = w$, we get an equation in $w[i]$ and $X[i+1]$ of the form

$$w[i] + [f \otimes 1, X[i+1]] = A,$$

where $A \in \mathfrak{g}_i \otimes \mathcal{V}(\mathfrak{p})$ is certain complicated expression, involving the adjoint action of $X[j+1]$, with $j < i$, on ∂ , q and $f \otimes 1$, which is homogeneous of $\text{ad } x$ -eigenvalue $\delta_x(A) = i$, and of conformal weight $\text{wt}(A) = 1$. Since $\mathfrak{g}_i = [f, \mathfrak{g}_{i+1}] \oplus V_i$, and since $\text{ad } f$ restricts to a bijection $\mathfrak{g}_{i+1} \xrightarrow{\sim} [f, \mathfrak{g}_{i+1}]$, the above equation determines uniquely $X[i+1] \in \mathfrak{g}_{i+1} \otimes \mathcal{V}(\mathfrak{p})$ (note that $\mathfrak{n} \cap \mathfrak{g}_{i+1} = \mathfrak{g}_{i+1}$ for $i \geq 0$) and $w[i] \in V_i \otimes \mathcal{V}(\mathfrak{p})$. Moreover, since $\text{wt}(A) = 1$, we also get that, necessarily, $\text{wt}(X[i+1]) = 0$ and $\text{wt}(w[i]) = 1$. This proves part (a).

For part (b), consider first the homogeneous components of $X \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ and $w \in V \otimes \mathcal{V}(\mathfrak{p})$ of degree 0, with respect to the polynomial degree decomposition (3.20): $X(0) \in \mathfrak{n} \otimes \mathbb{F} \simeq \mathfrak{n}$ and $w(0) \in V \otimes \mathbb{F} \simeq V$. Clearly, $\deg(f \otimes 1) = 0$ and $\deg(q) = 1$. By looking at the terms of polynomial degree 0 in both sides of the equation $q^X = w$, we get

$$(e^{\text{ad } X(0)} - 1)f = w(0).$$

With an inductive argument on the $\text{ad } x$ -eigenvalues, similar to the one use in the proof of (a), it is not hard to show that, necessarily, $X(0) = 0$ and $w(0) = 0$. Next, we study the homogeneous components of polynomial degree 1: $X(1) \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})(1) = \mathfrak{n} \otimes \mathbb{F}[\partial]\mathfrak{p}$ and $w \in V \otimes \mathcal{V}(\mathfrak{p})(1) = V \otimes \mathbb{F}[\partial]\mathfrak{p}$. By looking at the terms of polynomial degree 1 in both sides of the equation $q^X = w$, we get

$$w(1) + [f \otimes 1, X(1)] = q - X(1)', \quad (3.25)$$

By definition, $w(1) = \sum_{j \in J} v^j \otimes w_j(1)$, and $q = \sum_{j \in P} v^j \otimes v_j$. Equation (3.25) thus implies that

$$w_j(1) - v_j \in \partial \mathbb{F}[\partial]\mathfrak{p},$$

namely w_j admits the decomposition as in (3.24). The conditions on the conformal weights of the elements w_j and g_j immediately follow from the fact that $\Delta(w) = 1$.

Finally, we prove part (c). First, we prove that the elements $\{w_j\}_{j \in J}$ are differentially algebraically independent, that is they generate a differential polynomial algebra. For this, introduce in $\mathcal{V}(\mathfrak{p})$ the differential polynomial degree $\text{dd}(v_j^{(n)}) = n+1$ for every basis element $v_j \in \mathfrak{p}$ and $n \in \mathbb{Z}_+$. Suppose, by contradiction, that $P(w_1, \dots, w_r) = \sum w_{i_1}^{(n_1)} \dots w_{i_s}^{(n_s)} = 0$ is a non trivial differential polynomial relation among the w_j 's. If we let P_0 be the homogeneous component of P of minimal differential polynomial degree, then this relation can be written as $P_0(v_1, \dots, v_r) + \text{stuff of higher differential polynomial degree} = 0$. Hence $P_0(v_1, \dots, v_r) = 0$, contradicting the fact that the elements $v_1, \dots, v_r \in \mathfrak{g}^f \subset \mathfrak{p}$ are differentially algebraically independent in $\mathcal{V}(\mathfrak{p})$.

Next, we prove that all the coefficients w_j , $j \in J$, lie in \mathcal{W} . In view of Corollary 3.10, this is equivalent to prove that the differential polynomials $w_j \in \mathcal{V}(\mathfrak{p})$ are gauge invariant: $w_j(q^A) = w_j$ for every $A \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$. First, note that, obviously, $w(q^A) = \sum_{j \in J} v^j \otimes w_j(q^A)$ lies in $V \otimes \mathcal{V}(\mathfrak{p})$. On the other hand, by definition of gauge transformation, we have

$$w(q^A) = q^{X(q^A)}(q^A) = e^{\text{ad } X(q^A)}(\partial + f \otimes 1 + q^A) - \partial - f \otimes 1 = e^{\text{ad } X(q^A)} e^{\text{ad } A}(\partial + f \otimes 1 + q) - \partial - f \otimes 1.$$

By the Baker-Campbell-Hausdorff formula, there exists $\tilde{A} \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ such that $e^{\text{ad } X(q^A)} e^{\text{ad } A} = e^{\text{ad } \tilde{A}}$. Hence, the above equation reads $w(q^A) = q^{\tilde{A}} \in V \otimes \mathcal{V}(\mathfrak{p})$. By the uniqueness of $X \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ and $w \in V \otimes \mathcal{V}(\mathfrak{p})$ in part (a), it follows that, necessarily, $\tilde{A} = X$ and $w(q^A) = w$, as we wanted.

To conclude the proof of part (b), we are left to show that all the elements of \mathcal{W} are differential polynomials in w_1, \dots, w_r . Indeed, if $g \in \mathcal{W}$, then by Corollary 3.10 it is gauge invariant. Hence, in particular, $g = g(q^X) = g(w)$, namely it is expressed as a differential polynomial in the elements w_j , $j \in J$ (here we are using the obvious fact that, if we write w in basis $\{q^i\}_{i \in P}$ of \mathfrak{m}^\perp as $\sum_{i \in P} q^i \otimes h_i$, the elements h_i are linear combination of the w_j 's). \square

Corollary 3.13. *The Poisson vertex algebra \mathcal{W} , with the λ -bracket $\{g_\lambda h\}_{z, \rho}$, is independent of the choice of the isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ for $z = 0$, and for arbitrary z , provided that $s \in \text{Ker}(\text{ad } \mathfrak{g}_{\geq \frac{1}{2}})$ is fixed.*

Proof. Let $\mathfrak{l}_1 \subset \mathfrak{l}_2 \subset \mathfrak{g}_{\frac{1}{2}}$ be isotropic subspaces and $\mathfrak{m}_1 \subset \mathfrak{n}_1$ and $\mathfrak{m}_2 \subset \mathfrak{n}_2$ the corresponding nilpotent subalgebras of \mathfrak{g} defined in (3.2). Then

$$\mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \mathfrak{n}_2 \subset \mathfrak{n}_1. \quad (3.26)$$

Let $I_i = (m - \kappa(f | m) \mid m \in \mathfrak{m}_i) \subset \mathcal{V}(\mathfrak{g})$ and $\widetilde{\mathcal{W}}_i \subset \mathcal{V}(\mathfrak{g})$ be defined as in (3.13), for $i = 1, 2$. Clearly, by (3.26), $I_1 \subset I_2$, from which follows easily that $\widetilde{\mathcal{W}}_1 \subset \widetilde{\mathcal{W}}_2$. Hence, by Remark 3.6, we have a differential algebra homomorphism $\varphi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$, where \mathcal{W}_i , $i = 1, 2$, is the classical \mathcal{W} -algebra corresponding to \mathfrak{l}_i . For $z = 0$, or, for arbitrary z , provided that $s = s_1 = s_2 \in \text{Ker}(\text{ad } \mathfrak{g}_{\geq \frac{1}{2}})$, this is a PVA homomorphism. Indeed, in this case, $\widetilde{\mathcal{W}}_1 \subset \widetilde{\mathcal{W}}_2$ is a Poisson vertex subalgebra.

We want to show that φ is a differential algebra isomorphism. Fix $\mathfrak{p}_1, \mathfrak{p}_2 \subset \mathfrak{g}$ be complementary spaces to \mathfrak{m}_1 and \mathfrak{m}_2 respectively. Hence, $\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{p}_1 = \mathfrak{m}_2 \oplus \mathfrak{p}_2$. By the arbitrariness of the choice of these complementary subspaces (see Remark 3.6), we may assume $\mathfrak{p}_1 \supset \mathfrak{p}_2$. Let us denote by ρ_i , $i = 1, 2$, the differential algebra homomorphism defined in (3.5) corresponding to \mathfrak{p}_i . We have a differential algebra homomorphism induced by the following diagram

$$\begin{array}{ccc} \mathcal{V}(\mathfrak{g}) & \xrightarrow{\rho_1} & \mathcal{V}(\mathfrak{p}_1) \\ \rho_2 \downarrow & \swarrow \tilde{\varphi} & \\ \mathcal{V}(\mathfrak{p}_2) & & \end{array} \quad (3.27)$$

(it exists since $\text{Ker } \rho_1 = I_1 \subset \text{Ker } \rho_2 = I_2$). It is easy to check that $\tilde{\varphi}$ is the differential algebra homomorphism induced by the projection map $\pi_{\mathfrak{p}_2} : \mathfrak{p}_1 \rightarrow \mathfrak{p}_2$, and the restriction to $\mathcal{W}_1 = \mathcal{V}(\mathfrak{p}_1)^{\mathbb{F}[\partial]^{\mathfrak{n}}} \subset \mathcal{V}(\mathfrak{p}_1)$ is the differential algebra homomorphism $\varphi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ constructed above. Let $q_i \in \mathfrak{m}_i^\perp \otimes \mathcal{V}(\mathfrak{p}_i)$, for $i = 1, 2$, as in (3.14). We note that the choice of the vector space V in Theorem 3.12 does not depend on the choice of the isotropic subspace \mathfrak{l} (indeed $\text{ad } f : \mathfrak{g}_{\frac{1}{2}} \rightarrow \mathfrak{g}_{-\frac{1}{2}}$ is an isomorphism). Hence, we may choose $V \subset \mathfrak{m}_i^\perp$ such that $\mathfrak{m}_i^\perp = [f, \mathfrak{n}_i] \oplus V$, for $i = 1, 2$. Since the restriction of the differential map φ to $\mathfrak{p}_1 \subset \mathcal{V}(\mathfrak{p}_2)$ is the projection map $\pi_{\mathfrak{p}_2}$, we also note that $(\mathbb{1} \otimes \varphi)q_1 = q_2$. Let $X \in \mathfrak{n}_1 \otimes \mathcal{V}(\mathfrak{p}_1)$ such that $q_1^X \in V \otimes \mathcal{V}(\mathfrak{p}_1)$. Then $(\mathbb{1} \otimes \varphi)q_1^X = q_2^{(\mathbb{1} \otimes \tilde{\varphi})X} \in V \otimes \mathcal{V}(\mathfrak{p}_2)$. By the uniqueness argument in the proof of Theorem 3.12, it follows that $(\mathbb{1} \otimes \varphi)X \in \mathfrak{n}_2 \otimes \mathcal{V}(\mathfrak{p}_2)$. By Theorem 3.12(b), φ maps the generators of \mathcal{W}_1 to the generators of \mathcal{W}_2 . Hence it is an isomorphism.

Finally, if we take $\mathfrak{l}_1 = 0$, it follows that the PVA structure obtained for $z = 0$, or, for arbitrary z , provided that $s = s_1 = s_2 \in \text{Ker}(\text{ad } \mathfrak{g}_{\geq \frac{1}{2}})$, on \mathcal{W}_2 does not depend on the choice of the isotropic subspace \mathfrak{l}_2 . \square

Remark 3.14. It is proved in [GG02] for finite \mathcal{W} -algebras, by a method not applicable in our setup, that they are independent on the choice of the isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$.

Definition 3.15. Let \mathcal{V} be a PVA. An element $L \in \mathcal{V}$ is called a *Virasoro field* of *central charge* $c \in \mathbb{F}$ if

$$\{L_\lambda L\} = (\partial + 2\lambda)L + c\lambda^3. \quad (3.28)$$

An element $a \in \mathcal{V}$ is called an *eigenfield* of *conformal weight* $\Delta_a \in \mathbb{F}$ if

$$\{L_\lambda a\} = (\partial + \Delta_a \lambda)a + o(\lambda^2). \quad (3.29)$$

It is called a *primary field* of conformal weight Δ_a if $\{L_\lambda a\} = (\partial + \Delta_a \lambda)a$.

Proposition 3.16. *Consider the PVA \mathcal{W} with λ -bracket $\{\cdot_\lambda \cdot\}_{z, \rho}$ defined by equation (3.8).*

(a) *We have the following element*

$$L = \rho \left(\frac{1}{2} \sum_{i \in I} u^i u_i + x' \right) \in \mathcal{W},$$

where, as usual, $\{u_i\}_{i \in I}$ is a basis of \mathfrak{g} , $\{u^i\}_{i \in I}$ is the dual basis with respect to κ , and ρ is the map (3.5), and the λ -bracket of L with itself is

$$\{L_\lambda L\}_{z,\rho} = (\partial + 2\lambda)L - \kappa(x \mid x)\lambda^3 + 2\kappa(f \mid s)z\lambda. \quad (3.30)$$

In particular, $L \in \mathcal{W}$ is Virasoro field for $z = 0$ (or for arbitrary z provided that $\kappa(f \mid s) = 0$) of central charge $c = -\kappa(x \mid x)$.

- (b) Assume that $\mathfrak{p} \subset \mathfrak{g}$ is compatible with the $\text{ad } x$ -eigenspace decomposition (3.1), and consider the generators $w_j = v_j + g_j \in \mathcal{W}$, $j \in J$ provided by Theorem 3.12(b), where $\{v_j\}_{j \in J}$ is a basis of \mathfrak{g}^f consisting of $\text{ad } x$ -eigenvectors: $[x, v_j] = (1 - \Delta_j)v_j$, $j \in J$ (with $\Delta_j \geq 1$). Then w_j is an L -eigenfield of conformal weight Δ_j for $z = 0$.
- (c) The PVA \mathcal{W} is graded, as an algebra, by conformal weights: $\mathcal{W} = \mathbb{F} \oplus \mathcal{W}\{1\} \oplus \mathcal{W}\{\frac{3}{2}\} \oplus \mathcal{W}\{2\} \oplus \dots$. Moreover, for $i = 1$ or $\frac{3}{2}$, $\mathcal{W}\{i\}$ is spanned over \mathbb{F} by the generators w_j such that $\Delta(w_j) = 1 + \delta_x(v^j) = i$, and all of them are primary elements for $z = 0$.

Proof. We first prove part (a) in the case when $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ is maximal isotropic, that is when $\mathfrak{m} = \mathfrak{n}$. It is straightforward to check that

$$L^\mathfrak{g} = \frac{1}{2} \sum_{i \in I} u^i u_i + x' \in \mathcal{V}(\mathfrak{g})$$

is a Virasoro element of the affine PVA $\mathcal{V}(\mathfrak{g})$ (with λ -bracket (3.4)) with central charge $-\kappa(x \mid x)$:

$$\{L_\lambda^\mathfrak{g} L^\mathfrak{g}\}_z = (\partial + 2\lambda)L^\mathfrak{g} - \kappa(x \mid x)\lambda^3 + z(\partial + 2\lambda)[x, s],$$

and that, for $z = 0$, $a \in \mathfrak{g}_j$ is an eigenfield of conformal weight $\Delta = 1 - j$ (primary provided that $\kappa(x \mid a) = 0$), more precisely:

$$\begin{aligned} \{a_\lambda L^\mathfrak{g}\}_z &= -ja' + (1 - j)a\lambda + \kappa(a \mid x)\lambda^2 + z[s, a] + z\kappa([s, a] \mid x)\lambda, \\ \{L_\lambda^\mathfrak{g} a\}_z &= (\lambda + \partial)a - ja\lambda - \kappa(a \mid x)\lambda^2 - z[s, a] + z\kappa([s, a] \mid x)\lambda. \end{aligned} \quad (3.31)$$

For $a \in \mathfrak{n} \cap \mathfrak{g}_j$, we have, by the definition (3.6) of the action of $\mathbb{F}[\partial]\mathfrak{n}$ on $\mathcal{V}(\mathfrak{p})$,

$$a_\lambda^\rho L = \rho\{a_\lambda L\}_0 = \rho\{a_\lambda L^\mathfrak{g}\}_0 = \rho\left(-ja' + (1 - j)a\lambda + \kappa(a \mid x)\lambda^2\right),$$

where in the second equality we used Lemma 3.1(b). Since $\mathfrak{n} \subset \mathfrak{g}_{\geq \frac{1}{2}}$, we have $\kappa(a \mid x) = 0$, and since, by assumption, $\mathfrak{m} = \mathfrak{n}$, we also have $\rho(a) = \kappa(f \mid a)$. Hence the RHS above is equal to $(1 - j)\kappa(f \mid a)\lambda$, which is clearly zero. Therefore, recalling the definition (3.7) of the space \mathcal{W} , we conclude that $L \in \mathcal{W}$. It remains to prove that $L \in \mathcal{W}$ satisfies equation (3.30). By definition of the PVA structure on \mathcal{W} (3.8), we have, using Lemma 3.2(c)

$$\{L_\lambda L\}_{z,\rho} = \rho\{L_\lambda L\}_z = \rho\{L^\mathfrak{g}_\lambda L^\mathfrak{g}\}_z,$$

which, recalling the definition (3.5) of the map ρ , gives (3.30).

The case of arbitrary isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ follows from the case $\mathfrak{l} = 0$ since the Virasoro field $L_{(\mathfrak{l}=0)} = \rho_{(\mathfrak{l}=0)}(L^\mathfrak{g}) \in \mathcal{W}(\mathfrak{l}=0)$ corresponds precisely, via the isomorphism $\varphi : \mathcal{W}(\mathfrak{l}=0) \rightarrow \mathcal{W}(\mathfrak{l} \neq 0) = \mathcal{W}$ defined by the commutative diagram (3.27), to $L = \rho(L^\mathfrak{g})$.

It is easy to see, using equation (3.31) and Lemma 3.2(b), that the restriction of the conformal weight (3.22) to \mathcal{W} coincides with the Virasoro conformal weight for the Virasoro element L introduced in part (a). Hence, part (b) immediately follows from Theorem 3.12(b).

For (c), one only needs to prove that all elements in $\mathcal{W}\{1\}$ and $\mathcal{W}\{\frac{3}{2}\}$ are primary fields. First, if $w_j \in \mathcal{W}_{\frac{3}{2}}$ we have $\{L_\lambda w_j\} = \partial w_j + \frac{3}{2}\lambda w_j + L_{(2)}w_j\lambda^2 + \dots$. But for $n \geq 2$ the element $L_{(n)}w_j \in \mathcal{W}$ has conformal weight $\Delta(L_{(n)}w_j) = 2 + \frac{3}{2} - n - 1 \leq \frac{1}{2}$. Hence it must be $L_{(n)}w_j = 0$, proving that w_j is primary. Finally, if $w_j \in \mathcal{W}_1$, then, by Theorem 3.12(b), it must be of the form $w_j = v_j + \sum_k a_k b_k$, with $v_j \in \mathfrak{g}_0^f = \mathfrak{g}^f \cap \mathfrak{g}_0$ and $a_k, b_k \in \mathfrak{p}_{\frac{1}{2}} = \mathfrak{p} \cap \mathfrak{g}_{\frac{1}{2}}$. But by equation (3.31), for $z = 0$ all the elements $v_j \in \mathfrak{g}_0^f$ and $a_k, b_k \in \mathfrak{g}_{\frac{1}{2}}$ are primary field with respect to the Virasoro element $L^\mathfrak{g}$ in $\mathcal{V}(\mathfrak{g})$ (we use the fact that $\kappa(v_j \mid x) = \frac{1}{2}\kappa(v_j \mid [e, f]) = 0$ for $v_j \in \mathfrak{g}_0^f$). Hence, by the Leibniz rule w_j is also a primary field with respect $L^\mathfrak{g} \in \mathcal{V}(\mathfrak{g})$ for $z = 0$, and therefore, by Lemma 3.2(c), w_j is a primary field with respect to $L \in \mathcal{W}$ (for $z = 0$). \square

Corollary 3.17. Assume that $\mathfrak{p} \subset \mathfrak{g}$ is compatible with the $\text{ad } x$ -eigenspace decomposition (3.1) (in particular $\mathfrak{g}^f \subset \mathfrak{p}$). Let $\{v_j\}_{j \in J}$ be a basis of \mathfrak{g}^f consisting of $\text{ad } x$ -eigenvectors: $[x, v_j] = (1 - \Delta_j)v_j$, $j \in J$ (with $\Delta_j \geq 1$). Let $\{\tilde{w}_j\}_{j \in J}$ be an arbitrary collection of elements of \mathcal{W} of the form $\tilde{w}_j = v_j + \tilde{g}_j$, where

$$\tilde{g}_j = \sum b_1^{(m_1)} \dots b_s^{(m_s)} \in \mathcal{V}(\mathfrak{p})\{\Delta_j\},$$

is a sum such of products of $\text{ad } x$ -eigenvectors $b_i \in \mathfrak{p}$, such that

$$(1 - \delta_x(b_1)) + \cdots + (1 - \delta_x(b_s)) + m_1 + \cdots + m_s = \Delta_j,$$

and $s + m_1 + \cdots + m_s > 1$. (Such a collection of vectors exists by Theorem 3.12(b)). Then:

- (a) The elements $\{\tilde{w}_j\}_{j \in J}$ form a set of generators for the algebra of differential polynomials \mathcal{W} .
- (b) For $\Delta_j = 1$ or $\frac{3}{2}$, the generators \tilde{w}_j are uniquely determined by the corresponding basis elements $v_j \in \mathfrak{g}_j^f$, in particular, we have $\tilde{w}_j = w_j$ from Theorem 3.12(b).

Proof. By Theorem 3.12(c), each element \tilde{w}_j is a differential polynomial in the generators $\{w_k\}_{k \in J}$ defined in Theorem 3.12(b). On the other hand, if we order the generators according to their increasing conformal weights, each \tilde{w}_j is equal to w_j plus a differential polynomial P_j in the elements w_k 's with $k < j$. Hence, each w_j can be expressed as a differential polynomial in the \tilde{w}_k 's, proving part (a). Part (b) follows from the same argument and the observation that the conformal weight of the generators of \mathcal{W} are greater than or equal to 1, so that $P_j = 0$ if $\Delta_j = 1$ or $\frac{3}{2}$. \square

3.5. Examples of classical \mathcal{W} -algebras.

Example 3.18 (Virasoro-Magri and Gardner-Faddeev-Zakharov PVAs). Let $\mathfrak{g} = \mathfrak{sl}_2$ with standard generators $f, h = 2x, e$ and fix $\kappa(a | b) = \text{Tr}(ab)$, for any $a, b \in \mathfrak{sl}_2$. With respect to the $\text{ad } x$ -eigenspaces decomposition (3.1), we have $\mathfrak{n} = \mathfrak{m} = \mathbb{F}e$, $\mathfrak{m}^\perp = \mathbb{F}h \oplus \mathbb{F}e$, $[f, \mathfrak{n}] = \mathbb{F}h \subset \mathfrak{m}^\perp$. We fix the subspace $\mathfrak{p} = \mathbb{F}h \oplus \mathbb{F}f \subset \mathfrak{sl}_2$ complementary to \mathfrak{m} , and the subspace $V = \mathbb{F}e \subset \mathfrak{m}^\perp$ complementary to $[f, \mathfrak{n}]$. Then the element $X = e \otimes (-\frac{h}{2}) \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ brings $q = \frac{1}{2}h \otimes h + e \otimes f \in \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$ to

$$q^X = e \otimes \left(\frac{h^2}{4} + \frac{h'}{2} + f \right) \in V \otimes \mathcal{V}(\mathfrak{p}).$$

By Theorem 3.12, the corresponding \mathcal{W} -algebra is, as differential algebra, equal to the algebra of differential polynomials $\mathcal{W} = \mathbb{F}[w, w', w'', \dots] \subset \mathcal{V}(\mathfrak{p})$, where

$$w = \frac{h^2}{4} + \frac{h'}{2} + f \in \mathcal{V}(\mathfrak{p}).$$

We note that $w = L = \rho(L^{\mathfrak{sl}_2})$ (see Proposition 3.16), namely it is a Virasoro field for the $z = 0$ λ -bracket. If we take $s = e \in \text{Ker}(\text{ad } \mathfrak{n})$, by an easy computation we obtain

$$\{w_\lambda w\}_{z, \rho} = (2\lambda + \partial)w - \frac{\lambda^3}{2} + 2z\lambda.$$

Hence, the two compatible Hamiltonian structures $H, K \in \mathcal{W}[\lambda]$ associated to this family of λ -brackets via (1.9) are $H(\lambda) = (\partial + 2\lambda)w - \frac{\lambda^3}{2}$, known as the *Virasoro-Magri* Hamiltonian structure (of central charge $c = -\frac{1}{2}$), and $K(\lambda) = -2\lambda$, known as the *Gardner-Faddeev-Zakharov* Hamiltonian structure (up to the factor -2).

Example 3.19. Let $\mathfrak{g} = \mathfrak{sl}_3$ and fix $\kappa(a | b) = \text{Tr}(ab)$ for $a, b \in \mathfrak{sl}_3$. Let $f \in \mathfrak{sl}_3$ be its principal nilpotent element. In the matrix realization it is $f = E_{21} + E_{32}$. We can extend f to an \mathfrak{sl}_2 -triple $(f, h = 2x, e)$, with $x = E_{11} - E_{33}$. In this case $\mathfrak{g}_{\frac{1}{2}} = 0$, and hence $\mathfrak{n} = \mathfrak{m} \subset \mathfrak{sl}_3$ is the nilpotent subalgebra of strictly upper triangular matrices. Its orthogonal complement \mathfrak{m}^\perp consists of all upper triangular matrices. We can fix $\mathfrak{p} \subset \mathfrak{sl}_3$ to be the subspace, complementary to \mathfrak{m} , consisting of all lower triangular matrices. A basis of \mathfrak{p} is $\{h_1 = E_{11} - E_{22}, h_2 = E_{22} - E_{33}, E_{21}, E_{31}, E_{32}\}$. Also, as a subspace $V \subset \mathfrak{m}^\perp$ complementary to $[f, \mathfrak{n}] \subset \mathfrak{m}^\perp$ we choose, for example, $V = \mathfrak{g}^e = \text{Ker}(\text{ad } e) = \mathbb{F}(E_{12} + E_{23}) \oplus \mathbb{F}E_{13}$. After a straightforward computation, one can find $X \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ such that $q^X = (E_{12} + E_{23}) \otimes w_1 + E_{13} \otimes w_2 \in V \otimes \mathcal{V}(\mathfrak{p})$. The answer is as follows: $X = E_{12} \otimes a + E_{23} \otimes b + E_{13} \otimes c$, where

$$a = -\frac{1}{3}(2h_1 + h_2), \quad b = -\frac{1}{3}(h_1 + 2h_2), \quad c = \frac{1}{2}(E_{32} - E_{21}) - \frac{1}{6}(h_1^2 - h_2^2 + h'_1 - h'_2),$$

and

$$\begin{aligned} w_1 &= \frac{1}{2}(E_{21} + E_{32}) + \frac{1}{6}(h_1^2 + h_1 h_2 + h_2^2) + \frac{1}{2}(h'_1 + h'_2), \\ w_2 &= E_{31} + \frac{1}{3}h_1(E_{21} - 2E_{32}) + \frac{1}{3}h_2(2E_{21} - E_{32}) + \frac{2}{27}(h_1^3 - h_2^3) + \frac{1}{9}h_1 h_2(h_1 - h_2) \\ &\quad + \frac{1}{2}(E'_{21} - E'_{32}) + \frac{1}{6}h_1(2h'_1 - h'_2) + \frac{1}{6}h_2(h'_1 - 2h'_2) + \frac{1}{6}(h''_1 - h''_2). \end{aligned}$$

It is easy to check that $w_1 = \frac{1}{2}L = \rho(L^{\mathfrak{sl}_3})$ (see Proposition 3.16). Hence, by Theorem 3.12, L and w_2 generate the algebra of differential polynomial \mathcal{W} . Letting $s = E_{13} \in \text{Ker}(\text{ad } n)$, we get the following formulas for the λ -bracket (3.9) on the generators of the classical \mathcal{W} -algebra

$$\begin{aligned} \{L_\lambda L\}_{z,\rho} &= (2\lambda + \partial)L - 2\lambda^3, \\ \{L_\lambda w_2\}_{z,\rho} &= (3\lambda + \partial)w_2 + 3z\lambda, \\ \{w_2_\lambda w_2\}_{z,\rho} &= \frac{1}{6}L(\lambda + \partial)L + \frac{1}{4}(\lambda + \partial)L^2 + \frac{1}{4}\lambda L^2 - \frac{1}{6}(\lambda + \partial)^3L - \frac{1}{6}\lambda^3L - \frac{1}{4}\lambda(\lambda + \partial)(2\lambda + \partial)L + \frac{1}{6}\lambda^5. \end{aligned}$$

In particular, w_2 is a primary field of conformal weight 3. The corresponding compatible Hamiltonian structures $H, K \in \text{Mat}_{2 \times 2} \mathcal{W}[\lambda]$ are

$$H(\lambda) = \begin{pmatrix} (2\lambda + \partial)L - 2\lambda^3 & (3\lambda + 2\partial)w_2 \\ (3\lambda + \partial)w_2 & \frac{1}{6}L(\lambda + \partial)L + \frac{1}{4}(\lambda + \partial)L^2 + \frac{1}{4}\lambda L^2 - \frac{1}{6}(\lambda + \partial)^3L - \frac{1}{6}\lambda^3L - \frac{1}{4}\lambda(\lambda + \partial)(2\lambda + \partial)L + \frac{1}{6}\lambda^5 \end{pmatrix}, \quad K(\lambda) = \begin{pmatrix} 0 & -3\lambda \\ -3\lambda & 0 \end{pmatrix}.$$

Example 3.20. For $\mathfrak{g} = \mathfrak{sl}_3$, with inner product $\kappa(a | b) = \text{Tr}(ab)$, consider the lowest root vector $f = E_{31} \in \mathfrak{sl}_3$. We can extend it to an \mathfrak{sl}_2 -triple $(f, h = 2x, e)$, with $x = \frac{1}{2}E_{11} - \frac{1}{2}E_{33}$, $e = E_{13}$. In this case $\mathfrak{g}_{\frac{1}{2}} = \mathbb{F}E_{12} \oplus \mathbb{F}E_{23}$. Let us choose $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ to be the maximal isotropic subspace $\mathfrak{l} = \mathbb{F}(E_{12} + E_{23}) = \mathfrak{l}^{\perp\omega}$. In this case, $\mathfrak{n} = \mathfrak{m} = \mathbb{F}(E_{12} + E_{23}) \oplus \mathbb{F}E_{13} \subset \mathfrak{sl}_3$, and its orthogonal complement \mathfrak{m}^\perp is generated by $E_{21} - E_{32}$ and all upper triangular matrices in \mathfrak{sl}_3 . We can fix the subspace $\mathfrak{p} \subset \mathfrak{sl}_3$, complementary to \mathfrak{m} , with basis $\{g = E_{12} - E_{23}, h_1 = E_{11} - E_{22}, h_2 = E_{22} - E_{33}, E_{21}, E_{31}, E_{32}\}$. Also in this case, a subspace $V \subset \mathfrak{m}^\perp$ complementary to $[f, \mathfrak{n}] \subset \mathfrak{m}^\perp$ is, for example, $V = \mathfrak{g}^e = \mathbb{F}(h_1 - h_2) \oplus \mathbb{F}E_{12} \oplus \mathbb{F}E_{23} \oplus \mathbb{F}E_{13}$. Hence, we can find $X = (E_{12} + E_{23}) \otimes a + E_{13} \otimes b \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ such that $q^X = E_{13} \otimes w_1 + E_{12} \otimes w_2 + E_{23} \otimes w_3 + (h_1 - h_2) \otimes w_4 \in V \otimes \mathcal{V}(\mathfrak{p})$. The answer is as follows:

$$a = -\frac{1}{2}g, \quad b = -\frac{1}{2}h_1 - \frac{1}{2}h_2,$$

and

$$\begin{aligned} w_1 &= E_{31} - \frac{3}{64}g^4 + \frac{1}{2}g(E_{21} - E_{32}) + \frac{1}{8}g^2(h_1 - h_2) + \frac{1}{4}(h_1 + h_2)^2 + \frac{1}{2}h'_1 + \frac{1}{2}h'_2, \\ w_2 &= E_{21} - \frac{1}{8}g^3 + \frac{1}{2}gh_1 + \frac{1}{2}g', \quad w_3 = E_{32} + \frac{1}{8}g^3 + \frac{1}{2}gh_2 + \frac{1}{2}g', \quad w_4 = -\frac{1}{8}g^2 + \frac{1}{6}(h_1 - h_2). \end{aligned}$$

It is not hard to check that $L = \rho(L_3^{\mathfrak{sl}}) = w_1 + 3w_4^2$ (see Proposition 3.16). Hence, by Theorem 3.12, \mathcal{W} is the algebra of differential polynomials in the generators L, w_2, w_3, w_4 . Letting $s = E_{12} + E_{23} \in \text{Ker}(\text{ad } n)$, we get the following formulas for the λ -brackets (3.9) on the generators of \mathcal{W} :

$$\begin{aligned} \{L_\lambda L\}_{z,\rho} &= (2\lambda + \partial)L - \frac{1}{2}\lambda^3, \\ \{L_\lambda w_2\}_{z,\rho} &= \left(\frac{3}{2}\lambda + \partial\right)w_2 + \frac{3}{2}z\lambda, \\ \{L_\lambda w_3\}_{z,\rho} &= \left(\frac{3}{2}\lambda + \partial\right)w_3 + \frac{3}{2}z\lambda, \\ \{L_\lambda w_4\}_{z,\rho} &= (\lambda + \partial)w_4, \\ \{w_2_\lambda w_2\}_{z,\rho} &= 0, \\ \{w_2_\lambda w_3\}_{z,\rho} &= -w_1 + 9w_4^2 - 3(2\lambda + \partial)w_4 + \lambda^2, \\ \{w_2_\lambda w_4\}_{z,\rho} &= \frac{1}{2}w_2 + \frac{1}{2}z, \\ \{w_3_\lambda w_3\}_{z,\rho} &= 0, \\ \{w_3_\lambda w_4\}_{z,\rho} &= -\frac{1}{2}w_3 - \frac{1}{2}z, \\ \{w_4_\lambda w_4\}_{z,\rho} &= \frac{1}{6}\lambda. \end{aligned}$$

We note that, with respect to the $z = 0$ λ -bracket, w_2 and w_3 are primary fields of conformal weight $\frac{3}{2}$, and w_4 is a primary field of conformal weight 1.

We can also consider $\mathfrak{l} = 0$. Then $\mathfrak{l}^{\perp\omega} = \mathfrak{g}_{\frac{1}{2}}$. Hence, \mathfrak{n} consists of all strictly upper triangular matrices and $\mathfrak{m} = \mathbb{F}E_{13} \subset \mathfrak{n}$. The orthogonal complement \mathfrak{m}^\perp is spanned by E_{21}, E_{32} and all upper triangular matrices in \mathfrak{sl}_3 . We can fix the subspace $\mathfrak{p} \subset \mathfrak{sl}_3$, complementary to \mathfrak{m} , with basis $\{E_{12}, E_{23}, h_1 = E_{11} -$

$E_{22}, h_2 = E_{22} - E_{33}, E_{21}, E_{31}, E_{32}$. Again, as a subspace $V \subset \mathfrak{m}^\perp$ complementary to $[f, \mathfrak{n}] \subset \mathfrak{m}^\perp$ we take $V = \mathfrak{g}^e = \mathbb{F}(h_1 - h_2) \oplus \mathbb{F}E_{12} \oplus \mathbb{F}E_{23} \oplus \mathbb{F}E_{13}$. In this case, we can find $X = E_{12} \otimes a + E_{23} \otimes b + E_{13} \otimes c \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ such that $q^X = E_{13} \otimes w_1 + E_{12} \otimes w_2 + E_{23} \otimes w_3 + (h_1 - h_2) \otimes w_4 \in V \otimes \mathcal{V}(\mathfrak{p})$. We get the following answer:

$$a = E_{23}, \quad b = -E_{12}, \quad c = -\frac{1}{2}h_1 - \frac{1}{2}h_2$$

and

$$\begin{aligned} w_1 &= E_{31} + E_{12}E_{21} + E_{23}E_{32} - \frac{3}{4}E_{12}^2E_{32}^2 + \frac{1}{4}(h_1 + h_2)^2 - \frac{1}{2}E_{12}E_{23}(h_1 - h_2) + \frac{1}{2}E_{23}E'_{12} \\ &\quad - \frac{1}{2}E_{12}E'_{23} + \frac{1}{2}h'_1 + \frac{1}{2}h'_2, \quad w_2 = E_{21} - E_{12}E_{23}^2 - E_{23}h_1 - E'_{23}, \\ w_3 &= E_{32} - E_{12}^2E_{23} + E_{12}h_2 + E'_{12}, \quad w_4 = \frac{1}{2}E_{12}E_{23} + \frac{1}{6}(h_1 - h_2). \end{aligned}$$

Letting $s = E_{13} \in \text{Ker}(\text{ad } \mathfrak{n})$, we get the following formulas for the λ -brackets (3.9) on the generators of \mathcal{W} :

$$\begin{aligned} \{L_\lambda L\}_{z,\rho} &= (2\lambda + \partial)L - \frac{1}{2}\lambda^3 + 2z\lambda, \\ \{L_\lambda w_2\}_{z,\rho} &= \left(\frac{3}{2}\lambda + \partial\right)w_2, \\ \{L_\lambda w_3\}_{z,\rho} &= \left(\frac{3}{2}\lambda + \partial\right)w_3, \\ \{L_\lambda w_4\}_{z,\rho} &= (\lambda + \partial)w_4, \\ \{w_2\lambda w_2\}_{z,\rho} &= 0, \\ \{w_2\lambda w_3\}_{z,\rho} &= -w_1 + 9w_4^2 - 3(2\lambda + \partial)w_4 + \lambda^2 - z, \\ \{w_2\lambda w_4\}_{z,\rho} &= \frac{1}{2}w_2, \\ \{w_3\lambda w_3\}_{z,\rho} &= 0, \\ \{w_3\lambda w_4\}_{z,\rho} &= -\frac{1}{2}w_3, \\ \{w_4\lambda w_4\}_{z,\rho} &= \frac{1}{6}\lambda. \end{aligned}$$

As stated in Corollary 3.13 we note that the Hamiltonian structure corresponding to $z = 0$ does not change for different choices of the isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ (but it does change for arbitrary z with the change of s).

4. DRINFELD-SOKOLOV HIERARCHIES IN THE NON-HOMOGENEOUS CASE

In this section we construct, using the Lenard-Magri scheme, an integrable hierarchy of Hamiltonian equations for the classical \mathcal{W} -algebra defined in Definition 3.3. We use the same setup and notation as in the previous section.

4.1. Reformulation of the Lenard-Magri scheme.

Proposition 4.1. *Letting $L(z) = \partial + q + (f + zs) \otimes 1 \in \mathbb{F}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}))$, we have, for $a \in \mathfrak{p}$ and $g \in \mathcal{V}(\mathfrak{p})$,*

$$(H - zK)(a \otimes g) = (\pi_{\mathfrak{m}^\perp} \otimes 1)[L(z), a \otimes g].$$

Proof. It follows immediately from (3.12). \square

The variational derivative (1.3) in the algebra of differential polynomials $\mathcal{V}(\mathfrak{p})$, denoted by $\frac{\delta}{\delta q} : \mathcal{V}(\mathfrak{p}) \rightarrow \mathfrak{p} \otimes \mathcal{V}(\mathfrak{p})$, is given by ($g \in \mathcal{V}(\mathfrak{p})$):

$$\frac{\delta g}{\delta q} = \sum_{i \in P} q_i \otimes \frac{\delta g}{\delta q_i} \in \mathfrak{p} \otimes \mathcal{V}(\mathfrak{p}). \quad (4.1)$$

Lemma 4.2. *If $g \in \mathcal{W} \subset \mathcal{V}(\mathfrak{p})$, then $\left[L(z), \frac{\delta g}{\delta q}\right] \in \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{g})$.*

Proof. Let $\{q_i\}_{i \in M}$ be a basis of \mathfrak{m} , so that, if $I = P \cup M$, $\{q_i\}_{i \in I}$ is a basis of \mathfrak{g} . In order to prove the lemma, we only need to show that

$$\kappa\left(\left[L(z), \frac{\delta g}{\delta q}\right] \middle| q_k \otimes 1\right) = 0 \quad \text{for all } k \in M.$$

By the definition (3.7) of the space $\mathcal{W} \subset \mathcal{V}(\mathfrak{p})$, we have $\rho\{q_k \lambda g\}_z = 0$ for all $k \in M$ (since $\mathfrak{m} \subset \mathfrak{n}$). By the skewsymmetry of the λ -bracket, and using the fact that ρ is a homomorphism of differential algebras, we thus get, using the Master Formula (1.9) and the definition (3.4) of the λ -bracket $\{\cdot \lambda \cdot\}_z$ on $\mathcal{V}(\mathfrak{g})$, that

$$0 = \rho \{g \lambda q_k\}_z|_{\lambda=0} = \sum_{i \in P} \left(\pi_{\mathfrak{p}}[q_i, q_k] + \kappa(f + zs \mid [q_i, q_k]) + \kappa(q_i \mid q_k) \partial \right) \frac{\delta g}{\delta q_i},$$

for every $k \in M$. On the other hand, it is not hard to check that the RHS above is equal to $\kappa\left(\left[L(z), \frac{\delta g}{\delta q}\right] \mid q_k \otimes 1\right)$, proving the claim. \square

Corollary 4.3. *If $g \in \mathcal{W} \subset \mathcal{V}(\mathfrak{p})$, then*

$$(H - zK)\left(\frac{\delta g}{\delta q}\right) = \left[L(z), \frac{\delta g}{\delta q}\right].$$

Proof. It is an immediate consequence of Proposition 4.1 and Lemma 4.2. \square

According to the Lenard-Magri scheme of integrability [Mag78] (see also [BDSK09]), in order to find an integrable hierarchy of bi-Hamiltonian equations in \mathcal{W} , we need to find a sequence of linearly independent local functionals $\int g_n \in \mathcal{W}/\partial\mathcal{W}$, $n \in \mathbb{Z}_+$, such that

$$\int \{g_0 \lambda p\}_{K, \rho}|_{\lambda=0} = 0, \quad \text{and} \quad \int \{g_n \lambda p\}_{H, \rho}|_{\lambda=0} = \int \{g_{n+1} \lambda p\}_{K, \rho}|_{\lambda=0}, \quad (4.2)$$

for every $p \in \mathcal{W}$. In this case it is not hard to prove that we get the corresponding integrable hierarchy of Hamiltonian equations (see (1.18)):

$$\frac{dp}{dt_n} = \{g_n \lambda p\}_{H, \rho}|_{\lambda=0}, \quad n \in \mathbb{Z}_+.$$

We can reformulate the Lenard-Magri recursion relation (4.2) in terms of the matrices H and K defined in (3.11). By (3.10), equation (4.2) reads

$$\int \sum_{i, j \in P} \frac{\delta p}{\delta q_j} K_{ji}(\partial) \frac{\delta g_0}{\delta q_i} = 0, \quad \int \sum_{i, j \in P} \frac{\delta p}{\delta q_j} H_{ji}(\partial) \frac{\delta g_n}{\delta q_i} = \int \sum_{i, j \in P} \frac{\delta p}{\delta q_j} K_{ji}(\partial) \frac{\delta g_{n+1}}{\delta q_i}, \quad p \in \mathcal{W}. \quad (4.3)$$

Equivalently, we can rewrite equation (4.3) in terms of the maps $H, K : \mathfrak{p} \otimes \mathcal{V}(\mathfrak{p}) \rightarrow \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$ defined in (3.12). For this, we consider the non-degenerate pairing $(\mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})) \times (\mathfrak{p} \otimes \mathcal{V}(\mathfrak{p})) \rightarrow \mathcal{V}(\mathfrak{p})/\partial\mathcal{V}(\mathfrak{p})$, defined by

$$a \otimes g, b \otimes h \mapsto \int \kappa(a \mid b) gh.$$

In terms of the dual bases $\{q_i\}_{i \in P}$, $\{q^i\}_{i \in P}$ of \mathfrak{p} and \mathfrak{m}^\perp respectively, we pair $\sum_{i \in P} q^i \otimes g_i$, $\sum_{j \in P} q_j \otimes h_j \mapsto \int \sum_{i \in P} g_i h_i$. Then, the Lenard-Magri recursion relation (4.3) can be rewritten as

$$\int \kappa\left(K\left(\frac{\delta g_0}{\delta q}\right) \mid \frac{\delta p}{\delta q}\right) = 0, \quad \int \kappa\left(H\left(\frac{\delta g_n}{\delta q}\right) - K\left(\frac{\delta g_{n+1}}{\delta q}\right) \mid \frac{\delta p}{\delta q}\right) = 0, \quad \text{for all } p \in \mathcal{W}, n \in \mathbb{Z}_+.$$

In terms of the generating series $\int g(z) = \sum_{n \in \mathbb{Z}_+} \int g_n z^{-n+N} \in \mathcal{W}/\partial\mathcal{W}((z^{-1}))$ ($N \in \mathbb{Z}$ is arbitrary) these relations can be equivalently rewritten, using Corollary 4.3, as

$$\int \kappa\left(\left[L(z), \frac{\delta g(z)}{\delta q}\right] \mid \frac{\delta p}{\delta q}\right) = 0 \quad \text{in } \mathcal{V}(\mathfrak{p})/\partial\mathcal{V}(\mathfrak{p})((z^{-1})), \quad p \in \mathcal{W}. \quad (4.4)$$

Here and below, we extend κ to a bilinear map $(\mathfrak{g}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p})) \times (\mathfrak{g}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p})) \rightarrow \mathcal{V}(\mathfrak{p})((z^{-1}))$ as in (2.1) and linearly in z .

4.2. Basic assumptions. In the remainder of the paper we will assume that $s \in \text{Ker}(\text{ad } \mathfrak{n}) \subset \mathfrak{g}$ is a homogeneous element with respect to the $\text{ad } x$ -eigenspace decomposition (3.1), and that the Lie algebra $\mathfrak{g}((z^{-1}))$ admits a decomposition

$$\mathfrak{g}((z^{-1})) = \text{Ker } \text{ad}(f + zs) \oplus \text{Im } \text{ad}(f + zs) \quad (4.5)$$

(as pointed out in Remark 2.2, this is equivalent to semisimplicity of the element $f + zs$ of the reductive Lie algebra $\mathfrak{g}((z^{-1}))$ over the field $\mathbb{F}((z^{-1}))$). Under the above assumptions, we will be able to construct, in the following sections, the desired series $\int g(z) \in \mathcal{W}/\partial\mathcal{W}((z^{-1}))$ solving (4.4), thus providing an integrable hierarchy of bi-Hamiltonian equations in \mathcal{W} .

We denote $\mathfrak{h} = \text{Ker ad}(f + zs) \subset \mathfrak{g}((z^{-1}))$. Then $\text{Im ad}(f + zs) = \mathfrak{h}^\perp$ is the orthogonal complement to \mathfrak{h} with respect to the non-degenerate symmetric invariant bilinear form $\kappa_0 : \mathfrak{g}((z^{-1})) \times \mathfrak{g}((z^{-1})) \rightarrow \mathbb{F}$ (the constant term of κ on $\mathfrak{g}((z^{-1}))$) given by

$$\kappa_0(a(z) \mid b(z)) = \sum_{i \in \mathbb{Z}} \kappa(a_i \mid b_{-i}),$$

for $a(z) = \sum_{i \in \mathbb{Z}} a_i z^{-i}$ and $b(z) = \sum_{i \in \mathbb{Z}} b_i z^{-i} \in \mathfrak{g}((z^{-1}))$.

4.3. Outline. The applicability of the Lenard-Magri scheme of integrability will be achieved, following the ideas of Drinfeld and Sokolov [DS85], in four steps:

1. In Section 4.4 we find $h(z) \in \mathfrak{h} \otimes \mathcal{V}(\mathfrak{p})$ such that $e^{\text{ad } U(z)}(L(z)) = \partial + (f + zs) \otimes 1 + h(z)$ for some $U(z) \in \mathfrak{g}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p})$.
2. In Section 4.5 we prove that, if $a(z) \in Z(\mathfrak{h})$, then $\int g(z) = \int \kappa(a(z) \otimes 1 \mid h(z)) \in (\mathcal{V}(\mathfrak{p})/\partial \mathcal{V}(\mathfrak{p}))((z^{-1}))$ solves the Lenard-Magri recursion condition (4.4).
3. In Section 4.6 we prove that $\int g(z)$ defined above lies in $(\mathcal{W}/\partial \mathcal{W})((z^{-1}))$ (namely, the coefficients of $g(z)$ lie in \mathcal{W} up to total derivatives).
4. Finally, in Section 4.7 we prove that the coefficients $\int g_n$ of the Laurent series $\int g(z)$ span an infinite-dimensional subspace of $\mathcal{W}/\partial \mathcal{W}$.

4.4. Step 1. We extend the gradation (3.1) of \mathfrak{g} to a gradation of $\mathfrak{g}((z^{-1}))$ by letting $f + zs$ be homogeneous of degree -1 . In other words, if s has $\text{ad } x$ -eigenvalue $m \geq 0$ (it is eigenvector by assumption, and it lies in the centralizer of e , hence it has non-negative eigenvalue), then we let z have degree $-m - 1$.

Lemma 4.4. (a) For $i \in \frac{1}{2}\mathbb{Z}$, let $\mathfrak{g}((z^{-1}))_i \subset \mathfrak{g}((z^{-1}))$ be the space of homogeneous elements of degree i . We have the decomposition

$$\widehat{\mathfrak{g}((z^{-1}))} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \widehat{\mathfrak{g}((z^{-1}))_i}, \quad (4.6)$$

where the direct sum is completed by allowing infinite series in positive degrees.

(b) If $U(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$, then we have a well defined Lie algebra automorphism $e^{\text{ad } U(z)}$ of the Lie algebra $\mathbb{F}\partial \ltimes (\mathfrak{g}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p}))$.

Proof. Let Δ be the maximal eigenvalue of $\text{ad } x$ in \mathfrak{g} . Since z has degree $-m - 1 < 0$, we have

$$\mathfrak{g}((z^{-1}))_i \subset \bigoplus_{-\frac{i}{m+1} - \frac{\Delta}{m+1} \leq n \leq -\frac{i}{m+1} + \frac{\Delta}{m+1}} \mathfrak{g}z^n \quad \text{and} \quad \mathfrak{g}z^n \subset \bigoplus_{-n(m+1) - \Delta \leq i \leq -n(m+1) + \Delta} \mathfrak{g}((z^{-1}))_i. \quad (4.7)$$

The decomposition (4.6) follows immediately by the above inclusions. Part (b) follows from (a). \square

Note that, since $f + zs$ is homogeneous in $\mathfrak{g}((z^{-1}))$, then $\mathfrak{h} = \text{Ker ad}(f + zs) \subset \mathfrak{g}((z^{-1}))$ and $\mathfrak{h}^\perp = \text{Im ad}(f + zs) \subset \mathfrak{g}((z^{-1}))$ are compatible with the decomposition (4.6): $\mathfrak{h} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{h}_i$ and $\mathfrak{h}^\perp = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{h}_i^\perp$, where $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}((z^{-1}))_i$ and $\mathfrak{h}_i^\perp = \mathfrak{h}^\perp \cap \mathfrak{g}((z^{-1}))_i$. In the following, for $k \in \frac{1}{2}\mathbb{Z}$ and a subspace $V = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} V_i$ compatible with the decomposition (4.6) (such as \mathfrak{h} or \mathfrak{h}^\perp), we denote $V_{>k} = \bigoplus_{i > k} V_i$.

Proposition 4.5. Let $r = \sum_{i \in P} q^i \otimes r_i \in \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$.

(a) There exist unique formal Laurent series $U(z) \in \mathfrak{h}_{>0}^\perp \otimes \mathcal{V}(\mathfrak{p})$ and $h(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$ such that

$$e^{\text{ad } U(z)}(\partial + (f + zs) \otimes 1 + r) = \partial + (f + zs) \otimes 1 + h(z). \quad (4.8)$$

Moreover, the coefficients of $U(z)$ and $h(z)$ are differential polynomials in r_1, \dots, r_k .

(b) An automorphism $e^{\text{ad } U(z)}$, with $U(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$, solving (4.8) for some $h(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$, is defined uniquely up to multiplication on the left by automorphisms of the form $e^{\text{ad } S(z)}$, where $S(z) \in \mathfrak{h}_{>0} \otimes \mathcal{V}(\mathfrak{p})$.

Proof. Let us write $U(z) = \sum_{i \geq \frac{1}{2}} U_i(z)$, where $U_i(z) \in \mathfrak{h}_i^\perp \otimes \mathcal{V}(\mathfrak{p})$, $i \geq \frac{1}{2}$, and $h(z) = \sum_{i \geq -\frac{1}{2}} h_i(z)$, where $h_i(z) \in \mathfrak{h}_i \otimes \mathcal{V}(\mathfrak{p})$, $i \geq -\frac{1}{2}$. We will determine $U_{i+1}(z) \in \mathfrak{h}_{i+1}^\perp \otimes \mathcal{V}(\mathfrak{p})$ and $h_i(z) \in \mathfrak{h}_i \otimes \mathcal{V}(\mathfrak{p})$, inductively on $i \geq -\frac{1}{2}$, by equating the homogeneous components of degree i in each sides of equation (4.8).

Recall that $\mathfrak{m}^\perp \subset \mathfrak{g}_{\geq -\frac{1}{2}}$. Equating the terms of degree $-\frac{1}{2}$ in both sides of (4.8), we get the equation

$$h_{-\frac{1}{2}}(z) + [(f + zs) \otimes 1, U_{\frac{1}{2}}(z)] = (\pi_{-\frac{1}{2}} \otimes 1)r \in \mathfrak{g}_{-\frac{1}{2}} \otimes \mathcal{V}(\mathfrak{p}),$$

where $\pi_{-\frac{1}{2}} : \mathfrak{g}((z^{-1})) \rightarrow \mathfrak{g}((z^{-1}))_{-\frac{1}{2}}$ denotes the projection on the component of degree $-\frac{1}{2}$. Since we have the decomposition $\mathfrak{g}_{-\frac{1}{2}} \subset \mathfrak{g}((z^{-1}))_{-\frac{1}{2}} = \mathfrak{h}_{-\frac{1}{2}} \oplus \mathfrak{h}_{-\frac{1}{2}}^\perp$, and since $\text{ad}(f + zs)$ restricts to a bijection

$\mathfrak{h}_{\frac{1}{2}}^\perp \xrightarrow{\sim} \mathfrak{h}_{-\frac{1}{2}}^\perp$, the above equation determines uniquely $h_{-\frac{1}{2}}(z) \in \mathfrak{h}_{-\frac{1}{2}} \otimes \mathcal{V}(\mathfrak{p})$ and $U_{\frac{1}{2}}(z) \in \mathfrak{h}_{\frac{1}{2}} \otimes \mathcal{V}(\mathfrak{p})$. Moreover, the coefficients of $h_{-\frac{1}{2}}(z)$ and $U_{\frac{1}{2}}(z)$ are obviously differential polynomials in r_1, \dots, r_k .

Next, suppose by induction that we determined all elements $U_{j+1}(z) \in \mathfrak{h}_{j+1}^\perp \otimes \mathcal{V}(\mathfrak{p})$ and $h_j(z) \in \mathfrak{h}_j \otimes \mathcal{V}(\mathfrak{p})$ for $j < i$, and that their coefficients are differential polynomials in r_1, \dots, r_k . Equating the terms of degree i in both sides of (4.8), we get an equation in $h_i(z)$ and $U_{i+1}(z)$ of the form

$$h_i(z) + [(f + zs) \otimes 1, U_{i+1}(z)] = A(z),$$

where $A(z) \in \mathfrak{g}((z^{-1}))_i \otimes \mathcal{V}(\mathfrak{p})$ is certain complicated (differential polynomial) expression involving all the elements $U_{j+1}(z)$ and $h_j(z)$ for $j < i$. As before, since $\mathfrak{g}((z^{-1}))_i = \mathfrak{h}_i \oplus \mathfrak{h}_i^\perp$, and since $\text{ad}(f + zs)$ restricts to a bijection $\mathfrak{h}_{i+1}^\perp \xrightarrow{\sim} \mathfrak{h}_i^\perp$, the above equation determines uniquely $h_i(z) \in \mathfrak{h}_i \otimes \mathcal{V}(\mathfrak{p})$ and $U_{i+1}(z) \in \mathfrak{h}_{i+1}^\perp \otimes \mathcal{V}(\mathfrak{p})$, and their coefficients are differential polynomials in r_1, \dots, r_k . This proves part (a).

In the proof of part (b) we follow the same argument as in the proof of Proposition 2.3(b). Let $U(z) \in \mathfrak{h}_{>0}^\perp \otimes \mathcal{V}(\mathfrak{p})$, $h(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$ be the unique solution of (4.8) given by part (a). Let also $\tilde{U}(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$, $\tilde{h}(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$ be some other solution of (2.2): $e^{\text{ad} \tilde{U}(z)}(\partial + (f + zs) \otimes 1 + r) = \partial + (f + zs) \otimes 1 + \tilde{h}(z)$. By the Baker-Campbell-Hausdorff [Ser92], there exists $S(z) = \sum_{i>0}^\infty S_i(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$ such that $e^{\text{ad} \tilde{U}(z)} e^{-\text{ad} U(z)} = e^{\text{ad} S(z)}$. To conclude the proof of (b), we need to show that $S(z) \in \mathfrak{h}_{>0} \otimes \mathcal{V}(\mathfrak{p})$. By construction, we have

$$\partial + (f + zs) \otimes 1 + \tilde{h}(z) = e^{\text{ad} S(z)}(\partial + (f + zs) \otimes 1 + h(z)). \quad (4.9)$$

Comparing the terms of degree $-\frac{1}{2}$ in both sides of the above equation, we get

$$\mathfrak{h}_{-\frac{1}{2}}^\perp \otimes \mathcal{V}(\mathfrak{p}) \ni [(f + zs) \otimes 1, S_{\frac{1}{2}}(z)] = h_{-\frac{1}{2}}(z) - \tilde{h}_{-\frac{1}{2}}(z) \in \mathfrak{h}_{-\frac{1}{2}} \otimes \mathcal{V}(\mathfrak{p}).$$

Since $\mathfrak{h}_{-\frac{1}{2}}^\perp \cap \mathfrak{h}_{-\frac{1}{2}} = 0$, we conclude that $\tilde{h}_{-\frac{1}{2}}(z) = h_{-\frac{1}{2}}(z)$ and $S_{\frac{1}{2}}(z) \in \mathfrak{h}_{\frac{1}{2}} \otimes \mathcal{V}(\mathfrak{p})$. Next, assuming by induction that $S_j(z) \in \mathfrak{h}_j \otimes \mathcal{V}(\mathfrak{p})$ for all $j < i$, and comparing the terms of degree i in both sides of equation (4.9), we easily get that $[(f + zs) \otimes 1, S_{i+1}(z)] \in (\mathfrak{h}_i^\perp \otimes \mathcal{V}(\mathfrak{p})) \cap (\mathfrak{h}_i \otimes \mathcal{V}(\mathfrak{p})) = 0$, namely $S_{i+1}(z) \in \mathfrak{h}_{i+1} \otimes \mathcal{V}(\mathfrak{p})$, as desired. \square

Consider the special case when $r = q \in \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$. In this case, equation (4.8) reads

$$L_0(z) := e^{\text{ad} U(z)}(L(z)) = \partial + (f + zs) \otimes 1 + h(z). \quad (4.10)$$

Proposition 4.5 states that there exist unique $U_0(z) \in \mathfrak{h}_{>0}^\perp \otimes \mathcal{V}(\mathfrak{p})$ and $h_0(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$ solving (4.10), and any other solution of (4.10) with $U(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$ and $h(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$ is obtained from the unique one $(U_0(z), h_0(z))$ by taking $e^{\text{ad} U(z)} = e^{\text{ad} S(z)} e^{\text{ad} U_0(z)}$ for some $S(z) \in \mathfrak{h}_{>0} \otimes \mathcal{V}(\mathfrak{p})$.

4.5. Step 2. Throughout this section, we let $U(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$ and $h(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$ be a solution of equation (4.10), and we fix an element $a(z) \in Z(\mathfrak{h})$, the center of $\mathfrak{h} \subset \mathfrak{g}((z^{-1}))$. (For example, $a(z) = f + zs \in Z(\mathfrak{h})$). We also let

$$\int g(z) = \int \kappa(a(z) \otimes 1 \mid h(z)) \in \mathcal{V}(\mathfrak{p})/\partial \mathcal{V}(\mathfrak{p})((z^{-1})). \quad (4.11)$$

The main result of the section will be Theorem 4.9 below, where we show that $\int g(z)$ solves the Lenard-Magri recursion equation (4.4). In the following Sections 4.6 and 4.7 we will then show that, in fact, $\int g(z)$ is independent of the choice of the solution $U(z), h(z)$ of (4.10), that it lies in $(\mathcal{W}/\partial \mathcal{W})((z^{-1}))$, and that its coefficients span an infinite-dimensional subspace of $\mathcal{W}/\partial \mathcal{W}$, thus completing the proof of the applicability of the Lenard-Magri scheme of integrability.

Before proving the main theorem, we need some preliminary results. First, as immediate consequence of Proposition 4.5, we have the following result.

Corollary 4.6. *We have $[L(z), e^{-\text{ad} U(z)}(a(z) \otimes 1)] = 0$.*

Proof. Since, by assumption, $a(z) \in Z(\mathfrak{h}) \subset \mathfrak{h}$, we have $[\partial + (f + zs) \otimes 1 + h(z), a(z) \otimes 1] = 0$. Using the fact that $e^{-\text{ad} U(z)}$ is an automorphism of the Lie algebra $\mathbb{F} \partial \ltimes (\mathfrak{g}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p}))$, we have $[L(z), e^{-\text{ad} U(z)}(a(z) \otimes 1)] = e^{-\text{ad} U(z)}[\partial + (f + zs) \otimes 1 + h(z), a(z) \otimes 1] = 0$. \square

Lemma 4.7. *For $a \otimes g \in \mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$ and $p \in \mathcal{W}$, we have*

$$\int \kappa\left([L(z), a \otimes g] \mid \frac{\delta p}{\delta q}\right) = \int \rho\{a \partial p\}_{z \rightarrow g}. \quad (4.12)$$

Proof. By the definition (4.1) of the variational derivative we have

$$\int \kappa \left([L(z), a \otimes g] \left| \frac{\delta p}{\delta q} \right. \right) = \int \sum_{i \in P} \frac{\delta p}{\delta q_i} \kappa([L(z), a \otimes g] | q_i \otimes 1).$$

On the other hand, using Master Formula (1.9) and integration by parts, we get

$$\int \rho \{a \partial p\}_{z \rightarrow g} = \int \sum_{i \in P} \frac{\delta p}{\delta q_i} \rho \{a \partial q_i\}_{z \rightarrow g}.$$

Hence, equation (4.12) follows immediately from equation (3.18) with $n = 1$. \square

Lemma 4.8. *We have*

$$\frac{\delta g(z)}{\delta q} = (\pi_{\mathfrak{p}} \otimes 1) \left(e^{-\text{ad } U(z)} (a(z) \otimes 1) \right) \in (\mathfrak{p} \otimes \mathcal{V}(\mathfrak{p}))((z^{-1})), \quad (4.13)$$

where the projection $\pi_{\mathfrak{p}} : \mathfrak{g} \rightarrow \mathfrak{p}$ is extended to $\mathfrak{g}((z^{-1}))$ in the obvious way.

Proof. The proof follows from a straightforward computation following the same steps as in the proof of Proposition 2.4. By the definition (4.1) of the variational derivative and the definition (4.11) of $\int g(z)$, we have

$$\begin{aligned} \frac{\delta g(z)}{\delta q} &= \sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \frac{\partial g(z)}{\partial q_i^{(m)}} = \sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa \left(a(z) \otimes 1 \left| \frac{\partial h(z)}{\partial q_i^{(m)}} \right. \right). \\ &= \sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa \left(a(z) \otimes 1 \left| \frac{\partial}{\partial q_i^{(m)}} \left(e^{\text{ad } U(z)} (L(z)) - \partial - (f + zs) \otimes 1 \right) \right. \right) \end{aligned} \quad (4.14)$$

In the last identity we used equation (4.10). We next expand $e^{\text{ad } U(z)}$ in power series. The first term of the expansion is, by the definition (3.14) of $q \in \mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})$ and the first completeness relation (3.3),

$$\sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa \left(a(z) \otimes 1 \left| \frac{\partial q}{\partial q_i^{(m)}} \right. \right) = \sum_{i \in P} \kappa(a(z) | q^i) q_i \otimes 1 = \pi_{\mathfrak{p}} a(z) \otimes 1. \quad (4.15)$$

By Lemma 2.5, all the other terms in the power series expansion of the RHS of (4.14) are

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa \left(a(z) \otimes 1 \left| \frac{\partial}{\partial q_i^{(m)}} (\text{ad } U(z))^k L(z) \right. \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa \left(a(z) \otimes 1 \left| \sum_{h=0}^{k-1} (\text{ad } U(z))^h \left(\text{ad } \frac{\partial U(z)}{\partial q_i^{(m)}} \right) (\text{ad } U(z))^{k-h-1} L(z) \right. \right. \\ &\quad \left. \left. + (\text{ad } U(z))^k \frac{\partial}{\partial q_i^{(m)}} (q + (f + zs) \otimes 1) - (\text{ad } U(z))^{k-1} \frac{\partial U(z)}{\partial q_i^{(m-1)}} \right) \right) \\ &= \sum_{h, k \in \mathbb{Z}_+} \frac{1}{(h+k+1)!} \sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa \left(a(z) \otimes 1 \left| (\text{ad } U(z))^h \left(\text{ad } \frac{\partial U(z)}{\partial q_i^{(m)}} \right) (\text{ad } U(z))^k L(z) \right. \right) \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{k!} (\pi_{\mathfrak{p}} \otimes 1) ((-\text{ad } U(z))^k (a(z) \otimes 1)) \\ &\quad - \sum_{k \in \mathbb{Z}_+} \frac{1}{(k+1)!} \sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa \left(a(z) \otimes 1 \left| (\text{ad } U(z))^k \frac{\partial U(z)}{\partial q_i^{(m-1)}} \right. \right). \end{aligned} \quad (4.16)$$

For the first and last terms in the RHS we just changed the summation indices, while for the second term we used the first completeness relation (3.3) and the invariance of the bilinear map κ . Combining (4.15) and the second term in the RHS of (4.16), we get $(\pi_{\mathfrak{p}} \otimes 1) (e^{-\text{ad } U(z)} (a(z) \otimes 1))$, which is the same as the RHS of (4.13). Hence, in order to complete the proof of the proposition, we are left to show that the first and last term in the RHS of (4.16) cancel out. The last term of the RHS of (4.16) can be rewritten as

$$- \sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa(a(z) \otimes 1 | A_{i, m-1}(z)), \quad (4.17)$$

where $A_{i,m}(z) = \sum_{k \in \mathbb{Z}_+} \frac{1}{(k+1)!} (\text{ad } U(z))^k \frac{\partial U(z)}{\partial q_i^{(m)}}$. On the other hand, by Lemma 2.6, the first term of the RHS of (4.16) is equal to

$$\sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^m \kappa(a(z) \otimes 1 \mid [A_{i,m}(z), e^{\text{ad } U(z)} L(z)]).$$

By equation (4.10), the invariance of the bilinear map κ and the assumption that $a(z)$ lies in the center of \mathfrak{h} , the above expression is equal to

$$\sum_{i \in P, m \in \mathbb{Z}_+} q_i \otimes (-\partial)^{m+1} \kappa(a(z) \otimes 1 \mid A_{i,m}(z)),$$

which, combined with (4.17), gives zero. \square

Theorem 4.9. *The formal Laurent series $\int g(z) \in (\mathcal{V}(\mathfrak{p})/\partial \mathcal{V}(\mathfrak{p}))((z^{-1}))$ in (4.11) solves the Lenard-Magri recursion equation (4.4).*

Proof. By (4.13) we have

$$\frac{\delta g(z)}{\delta q} = (\pi_{\mathfrak{p}} \otimes 1) \left(e^{-\text{ad } U(z)} (a(z) \otimes 1) \right) = e^{-\text{ad } U(z)} (a(z) \otimes 1) - (\pi_{\mathfrak{m}} \otimes 1) \left(e^{-\text{ad } U(z)} (a(z) \otimes 1) \right),$$

so that, by Corollary 4.6, we get

$$\left[L(z), \frac{\delta g(z)}{\delta q} \right] = - \left[L(z), (\pi_{\mathfrak{m}} \otimes 1) \left(e^{-\text{ad } U(z)} (a(z) \otimes 1) \right) \right].$$

Hence, (4.4) holds by (4.12) and the definition (3.7) of the space $\mathcal{W} \subset \mathcal{V}(\mathfrak{p})$ (recall that $\mathfrak{m} \subset \mathfrak{n}$). \square

4.6. Step 3. In this section we will show that the Laurent series $\int g(z)$ given by (4.11) has coefficients in $\mathcal{W}/\partial \mathcal{W}$ (Proposition 4.11 below).

Lemma 4.10. *The Laurent series $\int g(z)$ defined by (4.11) is independent of the choice of $U(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$, $h(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$, solving equation (4.10).*

Proof. Let $\tilde{U}(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$, $\tilde{h}(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$ be any other solution of equation (4.10), and let $\int \tilde{g}(z) = \int \kappa(a(z) \otimes 1 \mid \tilde{h}(z))$. By Proposition (4.5)(b) there exists $S(z) \in \mathfrak{h}_{>0} \otimes \mathcal{V}(\mathfrak{p})$ such that $e^{\text{ad } \tilde{U}(z)} = e^{\text{ad } S(z)} e^{\text{ad } U(z)}$. By Lemma 4.8, we then have

$$\frac{\delta \tilde{g}(z)}{\delta q} = (\pi_{\mathfrak{p}} \otimes 1) \left(e^{-\text{ad } U(z)} e^{-\text{ad } S(z)} (a(z) \otimes 1) \right) = (\pi_{\mathfrak{p}} \otimes 1) \left(e^{-\text{ad } U(z)} (a(z) \otimes 1) \right) = \frac{\delta g(z)}{\delta q}.$$

In the second equality we used the assumption that $a(z) \in Z(\mathfrak{h})$. Since in the algebra of differential polynomials $\mathcal{V}(\mathfrak{p})$ we have $\text{Ker} \left(\frac{\partial}{\partial q} \right) = \partial \mathcal{V}(\mathfrak{p}) \oplus \mathbb{F}$, we deduce that $\int \tilde{g}(z)$ and $\int g(z)$ differ at most by a constant. On the other hand, as explained in the proof of Lemma 4.12, the constant term $\tilde{h}(z)[0] \in \mathfrak{h} \otimes 1$ of $\tilde{h}(z)$ is always zero. Therefore, the constant term of $\int \tilde{g}(z)$ is zero as well. \square

Proposition 4.11. *We have $\int g(z) \in (\mathcal{W}/\partial \mathcal{W})((z^{-1}))$.*

Proof. Fix a subspace $V \subset \mathfrak{m}^\perp$ complementary to $[f, \mathfrak{n}] \subset \mathfrak{m}^\perp$ compatible with the direct sum decomposition (3.1), and let $X \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ and $w \in V \otimes \mathcal{V}(\mathfrak{p})$ be the unique element provided by Theorem 3.12(a). By Proposition 4.5(a) and Theorem 3.12(b), there exist unique $U_w(z) \in \mathfrak{h}_{>0}^\perp \otimes \mathcal{W}$ and $h_w(z) \in \mathfrak{h}_{>0} \otimes \mathcal{W}$, such that

$$e^{\text{ad } U_w(z)} (\partial + (f + zs) \otimes 1 + w) = \partial + (f + zs) \otimes 1 + h_w(z).$$

By the identity $w = q^X$, we can rewrite the above equation as

$$e^{\text{ad } U_w(z)} e^{-\text{ad } X} (\partial + (f + zs) \otimes 1 + q) = \partial + (f + zs) \otimes 1 + h_w(z).$$

Since $\mathfrak{n} \subset \mathfrak{g}((z^{-1}))_{>0}$, by the Baker-Campbell-Hausdorff formula there exists $\tilde{U}(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$ such that $e^{\text{ad } U_w(z)} e^{-\text{ad } X} = e^{\text{ad } \tilde{U}(z)}$. Hence, $\tilde{U}(z), h_w(z)$ is another solution of equation (4.10). By Lemma 4.10 we thus conclude that $\int g(z) = \int \kappa(a(z) \otimes 1 \mid h_w(z)) \in (\mathcal{W}/\partial \mathcal{W})((z^{-1}))$. \square

4.7. Step 4. Consider the Laurent series $\int g(z) = \sum_{n \in \mathbb{Z}_+} \int g_n z^{-n+N}$ defined in (4.11). In this section we prove that, if $a(z) \in Z(\mathfrak{h})$ does not lie in the center of $\mathfrak{g}((z^{-1}))$, then the local functionals $\{\int g_n\}_{n \in \mathbb{Z}_+}$ span an infinite-dimensional subspace of $\mathcal{W}/\partial\mathcal{W}$.

We start by computing explicitly the linear (as polynomial in $\mathcal{V}(\mathfrak{p})$) part of $\frac{\delta g(z)}{\delta q}$. Let $U(z) \in \mathfrak{h}^\perp \otimes \mathcal{V}(\mathfrak{p})$ and $h(z) \in \mathfrak{h} \otimes \mathcal{V}(\mathfrak{p})$ be the unique solution of equation (4.10). Let $U(z) = \sum_{k \in \mathbb{Z}_+} U(z)[k]$ be the decomposition of $U(z)$ according to the usual grading of the algebra of differential polynomials $\mathcal{V}(\mathfrak{p}) = S(\mathbb{F}[\partial]\mathfrak{p})$.

Lemma 4.12. *The linear component of $U(z)$ is:*

$$U(z)[1] = \sum_{n \in \mathbb{Z}_+} (-1)^n (\text{ad}(f + zs)^{-n-1} \otimes 1) (\pi_{\mathfrak{h}^\perp} \otimes 1) \partial^n q, \quad (4.18)$$

where $\text{ad}(f + zs)^{-1}$ denotes the inverse map of the bijection $\text{ad}(f + zs)|_{\mathfrak{h}^\perp} : \mathfrak{h}^\perp \rightarrow \mathfrak{h}^\perp$, and $\pi_{\mathfrak{h}^\perp} : \mathfrak{g}((z^{-1})) \rightarrow \mathfrak{h}^\perp$ denotes the projection onto \mathfrak{h}^\perp with kernel \mathfrak{h} .

Proof. We proceed as in Remark 2.7. First, we equate the homogeneous components of degree 0 (as polynomials in $\mathcal{V}(\mathfrak{p})$) in both sides of (4.10). We get

$$(e^{\text{ad } U(z)[0]} - 1)(f + zs) \otimes 1 = h(z)[0].$$

Since $\mathfrak{h} \cap \mathfrak{h}^\perp = 0$, it is not hard to prove, inductively on the grading (4.6), that $U(z)[0] = 0$ in $\mathfrak{h}^\perp \otimes 1$ and $h(z)[0] = 0$ in $\mathfrak{h} \otimes 1$. In fact, the same argument can be used to prove that, for any solution $U(z) \in \mathfrak{g}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p})$, $h(z) \in \mathfrak{h} \otimes \mathcal{V}(\mathfrak{p})$ of equation (4.10), we have $h(z)[0] = 0$ and $U(z)[0] \in \mathfrak{h} \otimes 1$ (a fact that was used in the proof of Lemma 4.10).

Next, equating the homogeneous components of degree 1, we get that $h(z)[1] = (\pi_{\mathfrak{h}} \otimes 1)q$, while $U(z)[1]$ satisfies the equation

$$[(f + zs) \otimes 1, U(z)[1]] = (\pi_{\mathfrak{h}^\perp} \otimes 1)q - U'(z)[1].$$

Let $U(z)[1] = \sum_{0 \neq i \in \frac{1}{2}\mathbb{Z}_+} U(z)[1]_i$ be the decomposition of $U(z)[1]$ according to the grading (4.6). We can solve recursively the above equation by equating terms of the same degree with respect to the grading (4.6). We find that, for $i \geq -\frac{1}{2}$ in $\frac{1}{2}\mathbb{Z}$,

$$U(z)[1]_{i+1} = \sum_{n=0}^{[i+\frac{1}{2}]} (-1)^n (\text{ad}(f + zs)^{-n-1} \otimes 1) (\pi_{\mathfrak{h}_{i-n}^\perp} \otimes 1) \partial^n q$$

where $[i]$ denotes the integer part of $i \in \frac{1}{2}\mathbb{Z}$, and $\pi_{\mathfrak{h}_i^\perp}$ is the projection on $\mathfrak{h}_i^\perp = \mathfrak{g}((z^{-1}))_i \cap \mathfrak{h}^\perp$. Equation (4.18) is obtained from the above equation after summing over all possible degrees $i \geq -\frac{1}{2}$, and using the identity $\sum_i (\pi_{\mathfrak{h}_i^\perp} \otimes 1)q = (\pi_{\mathfrak{h}^\perp} \otimes 1)q$. \square

Lemma 4.13. *The linear part of $\frac{\delta g(z)}{\delta q} \in (\mathfrak{p} \otimes \mathcal{V}(\mathfrak{p}))((z^{-1}))$, with respect to the usual differential polynomial grading in $\mathcal{V}(\mathfrak{p})$, is*

$$\frac{\delta g(z)}{\delta q}[1] = \sum_{n \in \mathbb{Z}_+} (-1)^n (\pi_{\mathfrak{p}} \otimes 1) (\text{ad}(f + zs)^{-n-1} \otimes 1) [a(z) \otimes 1, \partial^n q], \quad (4.19)$$

where, as before, $\pi_{\mathfrak{p}} : \mathfrak{g} \rightarrow \mathfrak{p}$ denotes the projection onto \mathfrak{p} with kernel \mathfrak{m} , and it is extended to $\mathfrak{g}((z^{-1}))$ in the obvious way.

Proof. By Lemma 4.8 and equation (4.18), the linear part of $\frac{\delta g(z)}{\delta q}$ is given by

$$\begin{aligned} \frac{\delta g(z)}{\delta q}[1] &= (\pi_{\mathfrak{p}} \otimes 1) [a(z) \otimes 1, U(z)[1]] \\ &= \sum_{n \in \mathbb{Z}_+} (-1)^n (\pi_{\mathfrak{p}} \otimes 1) [a(z) \otimes 1, (\text{ad}(f + zs)^{-n-1} \otimes 1) (\pi_{\mathfrak{h}^\perp} \otimes 1) \partial^n q]. \end{aligned}$$

Equation (4.19) follows from the facts that, since $a(z)$ is in the center of $\mathfrak{h} = \text{Ker ad}(f + zs)$, we have that $\text{ad}(f + zs)$ and $\text{ad } a(z)$ commute, and $\text{ad } a(z) \circ \pi_{\mathfrak{h}^\perp} = \text{ad } a(z)$. \square

Lemma 4.14. *If $X \in \mathfrak{h}^\perp$ is non zero, then $\pi_{\mathfrak{p}} \text{ad}(f + zs)^{-n-1} X$ is different from zero for infinitely many values of $n \in \mathbb{Z}_+$.*

Proof. First note that, if $A \in \mathfrak{m}((z^{-1})) \setminus \{0\}$, then $\text{ad}(f + zs)A = [f, A]$ (since $s \in \text{Ker ad}(\mathfrak{n})$ and $\mathfrak{m} \subset \mathfrak{n}$), and therefore, since $\text{ad } f : \mathfrak{g}_i \rightarrow \mathfrak{g}_{i-1}$ is injective for $i > 0$, we have that $\text{ad}(f + zs)^k A \notin \mathfrak{m}((z^{-1}))$ for some $0 \leq k \leq \Delta$, where Δ is the maximal $\text{ad } x$ eigenvalue in \mathfrak{g} . Then, for every $n \in \mathbb{Z}_+$ such that $\pi_{\mathfrak{p}} \text{ad}(f + zs)^{-n-1} X = 0$ we have that $\pi_{\mathfrak{p}} \text{ad}(f + zs)^{k-n-1} X \neq 0$ for some $0 \leq k \leq \Delta$. The claim follows. \square

Lemma 4.15. (a) Consider the decomposition of $a(z) \in Z(\mathfrak{h})$ with respect to the grading (4.6): $a(z) = \sum_{k \in \frac{1}{2}\mathbb{Z}} a_k(z)$, where $a_k(z) \in Z(\mathfrak{h})_k$. Suppose that the basis $\{q^i\}_{i \in P}$ of \mathfrak{m}^\perp is homogeneous with respect to the decomposition (3.1), and let $h(i)$ be the degree of q^i . Let also $\frac{\delta g(z)}{\delta q}[1] = \sum_{-K \leq k \in \frac{1}{2}\mathbb{Z}} \frac{\delta g(z)}{\delta q}[1]_k$ be the decomposition of $\frac{\delta g(z)}{\delta q}[1] \in \mathfrak{p}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p})$ according to (4.6). Then we have

$$\frac{\delta g(z)}{\delta q}[1]_k = \sum_{i \in P} \sum_{n=0}^{N+k+h(i)+1} A_{k,i,n} \otimes q_i^{(n)},$$

with “leading term”

$$A_{k,i,N+k+h(i)+1} = (-1)^{k+N+h(i)+1} \pi_{\mathfrak{p}} \text{ad}(f + zs)^{-k-N-h(i)-2} [a_{-N}(z), q^i].$$

- (b) Assume that $a(z)$ does not lie in the center of $\mathfrak{g}((z^{-1}))$. Let $-N \in \frac{1}{2}\mathbb{Z}$ be the minimal degree such that $a_{-N}(z) \notin Z(\mathfrak{g}((z^{-1})))$, and let $\bar{h} = h(\bar{i})$, where $\bar{i} \in P$ is such that $[a_{-N}(z), q^{\bar{i}}] \neq 0$. Then the leading terms $A_{k,\bar{i},N+k+\bar{h}+1}$ are non zero for infinitely many values of $k \in \frac{1}{2}\mathbb{Z}$.
- (c) In particular, for infinitely many values of the degree k , the elements $\frac{\delta g(z)}{\delta q}[1]_k$ are non zero and they have distinct differential orders in the variable $q_{\bar{i}} \in \mathcal{V}(\mathfrak{p})$.

Proof. From equation (4.19) we have

$$\frac{\delta g(z)}{\delta q}[1]_k = \sum_{i \in P} \sum_{n \in \mathbb{Z}_+} (-1)^n \pi_{\mathfrak{p}} \text{ad}(f + zs)^{-n-1} [a_{k-n-h(i)-1}(z), q^i] \otimes q_i^{(n)}.$$

Part (a) follows from the above equation and the fact that, by assumption, $[a_k(z), q^i]$ can be non zero only for $k \geq -N$. Part (b) follows from part (a) and Lemma 4.14. Part (c) is obvious. \square

Lemma 4.16. Let $a(z) \in Z(\mathfrak{h}) \setminus Z(\mathfrak{g}((z^{-1})))$. Let $\frac{\delta g(z)}{\delta q}[1] = \sum_{n \in \mathbb{Z}_+} \frac{\delta g_n}{\delta q}[1] z^{-n+N}$ be the expansion of $\frac{\delta g(z)}{\delta q}[1] \in \mathfrak{p}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p})$ in power series of z . Then, the coefficients $\frac{\delta g_n}{\delta q}[1]$ span an infinite dimensional subspace of $\mathfrak{p} \otimes \mathcal{V}(\mathfrak{p})$.

Proof. It follows from Lemma 4.15(c) and the relation (4.7) between the decompositions of $\mathfrak{g}((z^{-1}))$ in powers of z and with respect to the grading (4.6). \square

Corollary 4.17. Let $a(z) \in Z(\mathfrak{h}) \setminus Z(\mathfrak{g}((z^{-1})))$. Letting $\int g(z) = \sum_{n \in \mathbb{Z}_+} \int g_n z^{-n+N} \in (\mathcal{W}/\partial \mathcal{W})((z^{-1}))$, the coefficients $\int g_n$, $n \in \mathbb{Z}_+$, span an infinite-dimensional subspace of $\mathcal{W}/\partial \mathcal{W}$.

Proof. Obvious from Lemma 4.16. \square

4.8. Conclusion. We can summarize all the results obtained in the previous sections in the following

Theorem 4.18. Consider the setup of Section 3.1 and assume, as in Section 4.2, that $s \in \text{Ker}(\text{ad } \mathfrak{n})$ is a homogeneous element such that $\mathfrak{g}((z^{-1}))$ decomposes as in (4.5). Let $a(z) \in Z(\mathfrak{h}) \setminus Z(\mathfrak{g}((z^{-1})))$, and let $U(z) \in \mathfrak{g}((z^{-1}))_{>0} \otimes \mathcal{V}(\mathfrak{p})$ and $h(z) \in \mathfrak{h}_{>-1} \otimes \mathcal{V}(\mathfrak{p})$ be a solution of equation (4.10). Consider the differential subalgebra $\mathcal{W} \subset \mathcal{V}(\mathfrak{p})$ defined in (3.7), with the compatible PVA structures $\{\cdot, \lambda \cdot\}_{H,\rho}$ and $\{\cdot, \lambda \cdot\}_{K,\rho}$ defined in (3.10)–(3.11). Then, the coefficients of the Laurent series $\int g(z) = \sum_{n \in \mathbb{Z}_+} \int g_n z^{N-n}$ defined in (4.11) span an infinite-dimensional subspace of $\mathcal{W}/\partial \mathcal{W}$ and they satisfy the Lenard-Magri recursion conditions (4.2). Hence, they are in involution with respect to both H and K :

$$\{\int g_m, \int g_n\}_{H,\rho} = \{\int g_m, \int g_n\}_{K,\rho} = 0 \quad \text{for all } m, n \in \mathbb{Z}_+,$$

and they define an integrable hierarchy of bi-Hamiltonian equations, called the generalized Drinfeld-Sokolov hierarchy:

$$\frac{dp}{dt_n} = \{g_n \lambda p\}_{H,\rho} \Big|_{\lambda=0} = \{g_{n+1} \lambda p\}_{K,\rho} \Big|_{\lambda=0}, \quad p \in \mathcal{W}, n \in \mathbb{Z}_+.$$

4.9. Examples.

Example 4.19 (The KdV hierarchy). Let us consider the classical \mathcal{W} -algebra corresponding to the Lie algebra \mathfrak{sl}_2 constructed in Example 3.18 and consider $a(z) = f + zs$. We get $\int g_0 = \int w$ and the corresponding Hamiltonian equation is $\frac{dw}{dt_0} = w'$. The next integral of motion is $\int g_1 = -\int \frac{w^2}{4}$ and the corresponding Hamiltonian equation is the *Korteweg-de Vries equation*

$$\frac{dw}{dt_1} = \frac{1}{4}(w''' - 6ww').$$

Example 4.20 (The Boussinesq hierarchy). Let us consider the classical \mathcal{W} -algebra corresponding to the Lie algebra \mathfrak{sl}_3 and its principal nilpotent element $f = E_{12} + E_{23}$, constructed in Example 3.19. Letting $a(z) = (f + zs)^2$ (recall that we are working in the matrix realization), we get $\int g_0 = \int w_2$ and the corresponding system of Hamiltonian equations is

$$\begin{cases} L_t = 2w_2' \\ w_{2t} = -\frac{1}{6}L''' + \frac{2}{3}LL' \end{cases}.$$

Eliminating w_2 from the system we get that L satisfies the *Boussinesq equation*

$$L_{tt} = -\frac{1}{3}L^{(4)} + \frac{4}{3}(LL')'.$$

Example 4.21. Let us consider the classical \mathcal{W} -algebra corresponding to the Lie algebra \mathfrak{sl}_3 and its minimal nilpotent element $f = E_{31}$, constructed in Example 3.20. In both cases considered (namely the choice $\mathfrak{l} = 0$ or $\mathfrak{l} \neq 0$) the element $f + zs$, where $s = E_{13}$, is semisimple and we get an integrable hierarchy of bi-Hamiltonian equations by Theorem 4.18. For example, when \mathfrak{l} is maximal isotropic, letting $a(z) = f + zs$ we get $\int g_0 = \int (w_2 + w_3)$ and the corresponding system of Hamiltonian equations is

$$\begin{cases} L_t = \frac{1}{2}(w_2' + w_3') \\ w_{2t} = L - 12w_4^2 - 3w_4' \\ w_{3t} = -L + 12w_4^2 - 3w_4' \\ w_{4t} = \frac{1}{2}(w_2 - w_3). \end{cases}$$

This system of equations was first studied in [BD91] and is known as *fractional KdV system*. Eliminating L, w_2, w_3 from the system we get that w_4 satisfies the equation

$$w_4'' = -\frac{1}{3}w_{4ttt} - 8(w_4w_{4t})_t,$$

which, after rescaling, is the Boussinesq equation with the derivatives with respect to x and t exchanged.

4.10. Applicability of the integrability scheme for \mathfrak{gl}_n . It is natural to ask when the assumptions of Theorem 4.18 hold, so that the proposed scheme of integrability can be applied. In other words, given a reductive Lie algebra \mathfrak{g} , we want to know for which nilpotent elements $f \in \mathfrak{g}$ (extended to an \mathfrak{sl}_2 -triple $f, h = 2x, e$), we are able to find an isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ and a homogeneous element $s \in \text{Ker}(\text{ad } \mathfrak{n})$ (where $\mathfrak{n} = \mathfrak{l}^{\perp\omega} \oplus \mathfrak{g}_{\geq 1}$) such that $f + zs$ is a semisimple element in $\mathfrak{g}((z^{-1}))$.

It is not hard to find a general answer in the case of \mathfrak{gl}_n . In this case, the integrability scheme can be applied successfully for all nilpotent elements $f \in \mathfrak{gl}_n$ corresponding to the partitions of n of the following type:

- (a) $n = r + \dots + r + 1 + \dots + 1$,
- (b) or $n = r + (r - 1) + \dots + r + (r - 1) + 1 + \dots + 1$.

For partitions of n of type $n = r + r + \dots + r + \epsilon$, where $\epsilon = 0$ or 1 , we can choose s in $\text{Ker}(\text{ad } \mathfrak{g}_{>0})$ (that is with $\mathfrak{l} = 0$), such that the corresponding element $f + zs \in \mathfrak{gl}_n((z^{-1}))$ is regular, homogeneous, semisimple, which corresponds to integrable hierarchies of “type I”, [dGHM92, BdGHM93, FHM92]. Removing the assumptions that $f + zs$ be regular (namely considering “type II hierarchies”) we allow partitions of n with an arbitrary number of $+1$ ’s. Furthermore, if the partition of n contains copies of $r + (r - 1)$, we are forced to choose s in $\text{Ker}(\text{ad } \mathfrak{n})$, with \mathfrak{n} strictly included in $\mathfrak{g}_{>0}$, namely we need to choose a non-zero isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$. For partitions as in types (a) and (b) above, the corresponding homogeneous

semisimple element $f + zs \in \mathfrak{g}((z^{-1}))$ is

$$f + zs^{(x)} = \begin{pmatrix} \Lambda_{r,x}^{DS}(z) & & & & 0 \\ & \Lambda_{r,x}^{DS}(z) & & & \\ & & \ddots & & \\ & & & \Lambda_{r,x}^{DS}(z) & \\ & & & & 0 \\ 0 & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix},$$

where $x = a$ or b , and

$$\Lambda_{r,a}^{DS}(z) = \begin{pmatrix} 0 & & & z \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \\ & & & 1 & 0 \end{pmatrix}, \quad \Lambda_{r,b}^{DS} = \left(\begin{array}{cccc|cccc} 0 & & & & & & & z \\ 1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ \hline & & & z & 0 & & & \\ & & & & 1 & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \end{array} \right) \left. \begin{array}{l} \left. \begin{array}{l} \vdots \\ \vdots \\ \vdots \end{array} \right\} r \\ \left. \begin{array}{l} \vdots \\ \vdots \\ \vdots \end{array} \right\} r-1 \end{array} \right\}.$$

We point out that our restrictions on the nilpotent element $f \in \mathfrak{gl}_n$ are the same as those obtained in [FGMS95, FGMS96], where they constructed generalized Drinfeld-Sokolov integrable hierarchies associated to a graded element in a Heisenberg subalgebra of $\mathfrak{g}((z^{-1}))$.

As a final remark, it is not clear if it is possible to further modify the setup of the integrability scheme to include other types of nilpotent elements $f \in \mathfrak{gl}_n$. For example, for $f \in \mathfrak{gl}_6$ corresponding to the partition $6 = 4 + 2$, we have $\mathfrak{gl}_{\frac{1}{2}} = 0$ (hence $\mathfrak{l} = 0$) since the gradation is even, and one can show that there is no choice of $s \in \text{Ker}(\text{ad } \mathfrak{n})$, homogeneous in the $\text{ad } x$ -eigenspaces decomposition, for which $f + zs$ is a semisimple element of $\mathfrak{gl}_6((z^{-1}))$.

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